# REVIEW TEAM : 2015-16

## FOR CLASS XII

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Name</th>
<th>Designation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### GROUP LEADER

Dr. Vandita Kalra  
*Vice Principal*  
SKV, Moti Nagar

### TEAM MEMBERS

1. **Sh. Devender Kumar**  
   *RPVV Hari Nagar, New Delhi*  
   *(PGT Maths)*

2. **Dr. Arun Kumar**  
   *SBM SSS Shivaji Marg, New Delhi*  
   *(PGT Maths)*

3. **Sh. Ashok Kumar Gupta**  
   *GBSSS, SU Block, Pitam Pura, Delhi*  
   *(PGT Maths)*
CLASS XII (2015 - 2016)

MATHEMATICS

<table>
<thead>
<tr>
<th>Units</th>
<th>No. of Periods</th>
<th>Weightage (Marks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Relations and Functions</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>(ii) Algebra</td>
<td>50</td>
<td>13</td>
</tr>
<tr>
<td>(iii) Calculus</td>
<td>80</td>
<td>44</td>
</tr>
<tr>
<td>(iv) Vector and Three Dimensional Geometry</td>
<td>30</td>
<td>17</td>
</tr>
<tr>
<td>(v) Linear Programming</td>
<td>20</td>
<td>06</td>
</tr>
<tr>
<td>(vi) Probability</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td>240</td>
<td>100</td>
</tr>
</tbody>
</table>

Unit I : RELATIONS AND FUNCTIONS

1. Relations and Functions (15 Periods)
   Types of Relations : Reflexive, symmetric, transitive and equivalence relations. Functions. One to one and onto functions, composite functions, inverse of a function. Binary operations.

2. Inverse Trigonometric Functions (15 Periods)
   Definition, range, domain, principal value branches. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

Unit II : ALGEBRA

1. Matrices (25 Periods)
   Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices.
Operation on Matrices : Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. Determinants (25 Periods)

Determinant of a square matrix (up to 3 × 3 matrices), properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Unit III : CALCULUS

1. Continuity and Differentiability (20 Periods)


2. Applications of Derivatives (10 Periods)

Applications of Derivatives : Rate of change of bodies, increasing/decreasing functions, tangents and normals, use of derivatives in approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

3. Integrals (20 Periods)

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts. Evaluation of simple integrals of the following types and problems based on them.
\[
\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}
\]
\[
\int \frac{(px + q)}{ax^2 + bx + c} \, dx, \int \frac{(px + q)}{\sqrt{ax^2 + bx + c}} \, dx, \int \frac{a^2 + x^2}{dx}, \int \frac{x^2 - a^2}{dx},
\]
\[
\int \sqrt{ax^2 + bx + c} \, dx \text{ and } \int (px + q)\sqrt{ax^2 + bx + c} \, dx
\]

Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

4. Applications of the Integrals (15 Periods)

Applications in finding the area under simple curves, especially lines, circles/parabolas/ellipses (in standard form only), area between any of the two above said curves (the region should be clearly identifiable).

5. Differential Equations (15 Periods)

Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables, Solution of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

\[
\frac{dy}{dx} + py = q, \text{ where } p \text{ and } q \text{ are functions of } x \text{ or constants}
\]

\[
\frac{dx}{dy} + px = q, \text{ where } p \text{ and } q \text{ are functions of } y \text{ or constants}
\]

Unit IV: VECTORS AND THREE-DIMENSIONAL GEOMETRY

1. Vectors (15 Periods)

Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and applications of scalar (dot) product of vectors, vector (cross) product of vectors, scalar triple product of vectors.
2. Three-Dimensional Geometry (15 Periods)

Direction cosines and direction ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines, (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.

Unit V : LINEAR PROGRAMMING (20 Periods)

1. Linear Programming : Introduction, related terminology such as constraints, objective function, optimization. Different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded and unbounded) feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

Unit VI : PROBABILITY (30 Periods)

1. Probability

Conditional probability, Multiplication theorem on probability, independent events, total probability, Baye’s theorem, Random variable and its probability distribution, mean and variance of a random variable. Repeated independent (Bernoulli) trials and Binomial distribution.

**QUESTION-WISE BREAK UP**

<table>
<thead>
<tr>
<th>Type of Questions</th>
<th>Marks per Question</th>
<th>Total Number of Questions</th>
<th>Total Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>VSA</td>
<td>1</td>
<td>6</td>
<td>06</td>
</tr>
<tr>
<td>LA - I</td>
<td>4</td>
<td>13</td>
<td>52</td>
</tr>
<tr>
<td>LA - II</td>
<td>6</td>
<td>7</td>
<td>42</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>26</td>
<td>100</td>
</tr>
</tbody>
</table>
## QUESTION PAPER DESIGN

### CLASS - XII (2015-16)

- **Time**: 3 hours
- **Max. Marks**: 100

<table>
<thead>
<tr>
<th>Sl.No.</th>
<th>Typology of Questions</th>
<th>Very Short Answer (1 M)</th>
<th>Long Answer I (4 M)</th>
<th>Long Answer II (6 M)</th>
<th>Marks</th>
<th>% Weightage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Remembering – (Knowledge based Simple recall questions, to know specific facts, terms, concepts, principles, or theories; Identify, define, or recite, information)</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>20</td>
<td>20%</td>
</tr>
<tr>
<td>2.</td>
<td>Understanding – (Comprehension-to be familiar with meaning and to understand conceptually, interpret, compare, contrast, explain, paraphrase information)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>22</td>
<td>22%</td>
</tr>
<tr>
<td>3.</td>
<td>Application – (Use abstract information in concrete situation, to apply knowledge to new situations; Use given content to interpret a situation, provide an example, or solve a problem)</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>29</td>
<td>29%</td>
</tr>
<tr>
<td>4.</td>
<td>High Order Thinking Skills – (Analysis &amp; Synthesis-classify, compare, contrast, or differentiate between different pieces of information; Organise and/or integrate unique pieces of information from a variety of sources)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>15%</td>
</tr>
<tr>
<td>5.</td>
<td>Evaluation – (Appraise, judge, and/or justify the value or worth of a decision or outcome, or to predict outcomes based on values)</td>
<td>1+1 (values based)</td>
<td>1</td>
<td>14</td>
<td>14%</td>
<td></td>
</tr>
</tbody>
</table>

**TOTAL**

|                      | 6×1= | 13×4= | 7×6= | 6 | 52 | 42 | 100 | 100% |

[Class XII : Maths] [6]
## CONTENTS

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Relations and Functions</td>
<td>9</td>
</tr>
<tr>
<td>2.</td>
<td>Inverse Trigonometric Functions</td>
<td>17</td>
</tr>
<tr>
<td>3 &amp; 4.</td>
<td>Matrices and Determinants</td>
<td>23</td>
</tr>
<tr>
<td>5.</td>
<td>Continuity and Differentiation</td>
<td>39</td>
</tr>
<tr>
<td>6.</td>
<td>Applications of Derivatives</td>
<td>47</td>
</tr>
<tr>
<td>7.</td>
<td>Integrals</td>
<td>61</td>
</tr>
<tr>
<td>8.</td>
<td>Applications of Integrals</td>
<td>84</td>
</tr>
<tr>
<td>9.</td>
<td>Differential Equations</td>
<td>89</td>
</tr>
<tr>
<td>10.</td>
<td>Vectors</td>
<td>100</td>
</tr>
<tr>
<td>11.</td>
<td>Three-Dimensional Geometry</td>
<td>110</td>
</tr>
<tr>
<td>12.</td>
<td>Linear Programming</td>
<td>121</td>
</tr>
<tr>
<td>13.</td>
<td>Probability</td>
<td>126</td>
</tr>
</tbody>
</table>

*Model Papers* 135
Chapter 1

Relations and Functions

Important Points to Remember

- Relation $R$ from a set $A$ to a set $B$ is subset of $A \times B$.
- $A \times B = \{(a, b) : a \in A, b \in B\}$.
- If $n(A) = r$, $n(B) = s$ from set $A$ to set $B$ then $n(A \times B) = rs$.
  and number of relations $= 2^{rs}$
- $\phi$ is also a relation defined on set $A$, called the void (empty) relation.
- $R = A \times A$ is called universal relation.
- Reflexive Relation: Relation $R$ defined on set $A$ is said to be reflexive iff $(a, a) \in R \ \forall \ a \in A$
- Symmetric Relation: Relation $R$ defined on set $A$ is said to be symmetric iff $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b, \in A$
- Transitive Relation: Relation $R$ defined on set $A$ is said to be transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in A$
- Equivalence Relation: A relation defined on set $A$ is said to be equivalence relation iff it is reflexive, symmetric and transitive.
- One-One Function: $f : A \rightarrow B$ is said to be one-one if distinct elements in $A$ have distinct images in $B$. i.e. $\forall \ x_1, x_2 \in A$ such that $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
  OR
  $\forall \ x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$
  $\Rightarrow x_1 = x_2$
  One-one function is also called injective function.
Onto function (surjective) : A function \( f : A \to B \) is said to be onto iff \( R_f = B \) i.e. \( \forall b \in B, \) there exists \( a \in A \) such that \( f(a) = b \)

A function which is not one-one is called many-one function.

A function which is not onto is called into function.

Bijective Function : A function which is both injective and surjective is called bijective function.

Composition of Two Functions : If \( f : A \to B, \ g : B \to C \) are two functions, then composition of \( f \) and \( g \) denoted by \( gof \) is a function from \( A \) to \( C \) given by, \( (gof)(x) = g(f(x)) \ \forall \ x \in A \)

Clearly \( gof \) is defined if Range of \( f \subset \) domain of \( g \). Similarly \( fog \) can be defined.

Invertible Function : A function \( f : X \to Y \) is invertible iff it is bijective.

If \( f : X \to Y \) is bijective function, then function \( g : Y \to X \) is said to be inverse of \( f \) iff \( fog = I_y \) and \( gof = I_x \)

when \( I_x, I_y \) are identity functions.

\( g \) is inverse of \( f \) and is denoted by \( f^{-1} \).

Binary Operation : A binary operation \( * \) defined on set \( A \) is a function from \( A \times A \to A \).

\( * \) \((a, b)\) is denoted by \( a * b \).

Binary operation \( * \) defined on set \( A \) is said to be commutative iff \( a * b = b * a \ \forall \ a, b \in A \).

Binary operation \( * \) defined on set \( A \) is called associative iff \( a * (b * c) = (a * b) * c \ \forall \ a, b, c \in A \)

If \( * \) is Binary operation on \( A \), then an element \( e \in A \) is said to be the identity element iff \( a * e = e * a = a \ \forall \ a \in A \)

Identity element is unique.

If \( * \) is Binary operation on set \( A \), then an element \( b \in A \) is said to be inverse of \( a \in A \) iff \( a * b = b * a = e \)

Inverse of an element, if it exists, is unique.
**VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)**

1. If A is the set of students of a school then write, which of following relations are Universal, Empty or neither of the two.
   
   \[ R_1 = \{(a, b) : a, b \text{ are ages of students and } |a - b| \geq 0\} \]
   
   \[ R_2 = \{(a, b) : a, b \text{ are weights of students, and } |a - b| < 0\} \]
   
   \[ R_3 = \{(a, b) : a, b \text{ are students studying in same class}\} \]

2. Is the relation \( R \) in the set \( A = \{1, 2, 3, 4, 5\} \) defined as
   
   \[ R = \{(a, b) : b = a + 1\} \]
   
   reflexive?

3. If \( R \), is a relation in set \( N \) given by
   
   \[ R = \{(a, b) : a = b - 3, b > 5\}, \]
   
   then does element \((5, 7) \in R?\)

4. If \( f : \{1, 3\} \to \{1, 2, 5\} \) and \( g : \{1, 2, 5\} \to \{1, 2, 3, 4\} \) be given by
   
   \( f = \{(1, 2), (3, 5)\}, \ g = \{(1, 3), (2, 3), (5, 1)\}, \)
   
   write gof.

5. Let \( g, f : R \to R \) be defined by
   
   \[ g(x) = \frac{x + 2}{3}, f(x) = 3x - 2. \]
   
   write fog \((x)\)

6. If \( f : R \to R \) defined by
   
   \[ f(x) = \frac{2x - 1}{5} \]
   
   be an invertible function, write \( f^{-1}(x)\).

7. If \( f(x) = \frac{x}{x + 1} \forall x \neq -1 \), write \( fo f(x)\).

8. Let * be a Binary operation defined on \( R \), then if
   
   \[ (i) \quad a * b = a + b + ab, \text{ write } 3 * 2 \]
   
   \[ (ii) \quad a * b = \frac{(a + b)^2}{3}, \text{ write } (2 * 3) * 4. \]
9. If \( n(A) = n(B) = 3 \), then how many bijective functions from \( A \) to \( B \) can be formed?

10. If \( f(x) = x + 1 \), \( g(x) = x - 1 \), then \((gof)(3) = ?\)

11. Is \( f : N \rightarrow N \) given by \( f(x) = x^2 \) one-one? Give reason.

12. If \( f : R \rightarrow A \), given by
\[
f(x) = x^2 - 2x + 2\]
is onto function, find set \( A \).

13. If \( f : A \rightarrow B \) is bijective function such that \( n(A) = 10 \), then \( n(B) = ? \)

14. If \( f : R \rightarrow R \) defined by \( f(x) = \frac{x - 1}{2} \), find \((fof)(x)\)

15. \( R = \{(a, b) : a, b \in N, a \neq b \text{ and } a \text{ divides } b\} \). Is \( R \) reflexive? Give reason

16. Is \( f : R \rightarrow R \), given by \( f(x) = |x - 1| \) one-one? Give reason

17. \( f : R \rightarrow B \) given by \( f(x) = \sin x \) is onto function, then write set \( B \).

18. If \( f(x) = \log \left( \frac{1 + x}{1 - x} \right) \), show that \( f \left( \frac{2x}{1 + x^2} \right) = 2f(x) \).

19. If \( * \) is a binary operation on set \( Q \) of rational numbers given by \( a * b = \frac{ab}{5} \)
then write the identity element in \( Q \).

20. If \( * \) is Binary operation on \( N \) defined by \( a * b = a + ab \ \forall \ a, b \in N \), write the identity element in \( N \) if it exists.

**SHORT ANSWER TYPE QUESTIONS (4 Marks)**

21. Check the following functions for one-one and onto.

   (a) \( f : R \rightarrow R, \ f(x) = \frac{2x - 3}{7} \)

   (b) \( f : R \rightarrow R, \ f(x) = |x + 1| \)

   (c) \( f : R - \{2\} \rightarrow R, \ f(x) = \frac{3x - 1}{x - 2} \)

   (d) \( f : R \rightarrow [-1, 1], \ f(x) = \sin^2 x \)

22. Consider the binary operation \( * \) on the set \( \{1, 2, 3, 4, 5\} \) defined by \( a * b = \text{H.C.F. of } a \text{ and } b \). Write the operation table for the operation \( * \).
23. Let \( f : \mathbb{R} - \left[ \frac{-4}{3} \right] \rightarrow \mathbb{R} - \left[ \frac{4}{3} \right] \) be a function given by \( f(x) = \frac{4x}{3x + 4} \). Show that \( f \) is invertible with \( f^{-1}(x) = \frac{4x}{4 - 3x} \).

24. Let \( R \) be the relation on set \( A = \{ x : x \in \mathbb{Z}, 0 \leq x \leq 10 \} \) given by \( R = \{ (a, b) : (a - b) \) is divisible by 4 \}. Show that \( R \) is an equivalence relation. Also, write all elements related to 4.

25. Show that function \( f : A \rightarrow B \) defined as \( f(x) = \frac{3x + 4}{5x - 7} \) where \( A = R - \left\{ \frac{7}{5} \right\}, B = R - \left\{ \frac{3}{5} \right\} \) is invertible and hence find \( f^{-1} \).

26. Let \( * \) be a binary operation on \( \mathbb{Q} \) such that \( a * b = a + b - ab \).
   (i) Prove that \( * \) is commutative and associative.
   (ii) Find identify element of \( * \) in \( \mathbb{Q} \) (if it exists).

27. If \( * \) is a binary operation defined on \( R - \{0\} \) defined by \( a * b = \frac{2a}{b^2} \), then check \( * \) for commutativity and associativity.

28. If \( A = \mathbb{N} \times \mathbb{N} \) and binary operation \( * \) is defined on \( A \) as \( (a, b) * (c, d) = (ac, bd) \).
   (i) Check \( * \) for commutativity and associativity.
   (ii) Find the identity element for \( * \) in \( A \) (if it exists).

29. Show that the relation \( R \) defined by \( (a, b) R(c, d) \iff a + d = b + c \) on the set \( \mathbb{N} \times \mathbb{N} \) is an equivalence relation.

30. Let \( * \) be a binary operation on set \( \mathbb{Q} \) defined by \( a * b = \frac{ab}{4} \), show that
   (i) 4 is the identity element in \( \mathbb{Q} \).
   (ii) Every non-zero element of \( \mathbb{Q} \) is invertible with \( a^{-1} = \frac{16}{a}, \forall a \in \mathbb{Q} - \{0\} \).

31. Show that \( f : R_+ \rightarrow R_+ \) defined by \( f(x) = \frac{1}{2x} \) is bijective where \( R_+ \) is the set of all non-zero positive real numbers.
32. Let $A = \{1, 2, 3, ..., 12\}$ and $R$ be a relation in $A \times A$ defined by $(a, b) \sim (c, d)$ if $ad = bc \ \forall \ (a, b), (c, d) \in A \times A$. Prove that $R$ is an equivalence relation. Also obtain the equivalence class $[(3, 4)]$.

33. If $\ast$ is a binary operation on $R$ defined by $a \ast b = a + b + ab$. Prove that $\ast$ is commutative and associative. Find the identity element. Also show that every element of $R$ is invertible except $-1$.

34. If $f, g : R \to R$ defined by $f(x) = x^2 - x$ and $g(x) = x + 1$ find $(fog)(x)$ and $(gof)(x)$. Are they equal?

35. $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, find $f^{-1}(x)$.

36. $f : R \to R, g : R \to R$ given by $f(x) = [x], g(x) = |x|$ then find

$$(fog)\left(\frac{-2}{3}\right) \text{ and } (gof)\left(\frac{-2}{3}\right).$$

**LONG ANSWER TYPE QUESTIONS (6 Marks)**

37. Let $N$ denote the set of all natural numbers and $R$ be the relation on $N \times N$ defined by $(a, b) \sim (c, d)$ if $ad (b + c) = bc (a + d)$. Show that $R$ is an equivalence relation.

38. Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to S$, where $S$ is the range of $f$, is invertible. Also find the inverse of $f$.

**ANSWERS**

1. $R_1$ : is universal relation.
   $R_2$ : is empty relation.
   $R_3$ : is neither universal nor empty.

2. No, $R$ is not reflexive.

3. $(5, 7) \notin R$

4. $gof = \{(1, 3), (3, 1)\}$

5. $(fog)(x) = x \ \forall \ x \in R$
6. \( f^{-1}(x) = \frac{5x + 1}{2} \)

7. \( (f \circ f)(x) = \frac{x}{2x + 1}, x \neq -\frac{1}{2} \)

8. (i) \( 3 \times 2 = 11 \)

(ii) \( \frac{1369}{27} \)

9. 6

10. 3

11. Yes, \( f \) is one-one \( \because \forall x_1, x_2 \in \mathbb{N} \Rightarrow x_1^2 = x_2^2 \).

12. \( A = [1, \infty) \) because \( R_f = [1, \infty) \)

13. \( n(B) = 10 \)

14. \( (f \circ f)(x) = \frac{x - 3}{4} \)

15. No, \( R \) is not reflexive \( \because (a, a) \notin R \) \( \forall a \in \mathbb{N} \)

16. \( f \) is not one-one function

\( \because f(3) = f(-1) = 2 \)

3 \( \neq -1 \) i.e. distinct elements have same images.

17. \( B = [-1, 1] \)

19. \( e = 5 \)

20. Identity element does not exists.

21. (a) Bijective

(b) Neither one-one nor onto.

(c) One-one, but not onto.

(d) Neither one-one nor onto.
22. 

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

24. Elements related to 4 are 0, 4, 8.

25. \( f^{-1}(x) = \frac{7x + 4}{5x - 3} \)

26. 0 is the identity element.

27. Neither commutative nor associative.

28. (i) Commutative and associative.

(ii) (1, 1) is identity in \( N \times N \)

32. \([3, 4] = \{(3, 4), (6, 8), (9, 12)\}\)

33. 0 is the identity element.

34. \((fog)(x) = x^2 + x\)

\((gof)(x) = x^2 - x + 1\)

Clearly, they are unequal.

35. \( f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2} \)

36. \((fog)\left(\frac{-2}{3}\right) = 0\)

\((gof)\left(\frac{-2}{3}\right) = 1\)

38. \( f^{-1}(y) = \frac{\sqrt{y - 6} - 3}{2} \)
CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

IMPORTANT POINTS

- \( \sin^{-1} x, \cos^{-1} x, \ldots \) etc. are angles.

- If \( \sin \theta = x \) and \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), then \( \theta = \sin^{-1} x \) etc.

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range (Principal Value Branch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1} x )</td>
<td>([-1, 1])</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
</tr>
<tr>
<td>( \cos^{-1} x )</td>
<td>([-1, 1])</td>
<td>([0, \pi])</td>
</tr>
<tr>
<td>( \tan^{-1} x )</td>
<td>(\mathbb{R})</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
</tr>
<tr>
<td>( \cot^{-1} x )</td>
<td>(\mathbb{R})</td>
<td>((0, \pi))</td>
</tr>
<tr>
<td>( \sec^{-1} x )</td>
<td>(\mathbb{R} - (-1, 1))</td>
<td>([0, \pi) - \left(\frac{\pi}{2}\right))</td>
</tr>
<tr>
<td>( \csc^{-1} x )</td>
<td>(\mathbb{R} - (-1, 1))</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}] - {0})</td>
</tr>
</tbody>
</table>

- \( \sin^{-1} \left( \sin x \right) = x \ \forall \ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \)

- \( \cos^{-1} \left( \cos x \right) = x \ \forall \ x \in [0, \pi] \) etc.

- \( \sin \left( \sin^{-1} x \right) = x \ \forall \ x \in [-1, 1] \)

- \( \cos \left( \cos^{-1} x \right) = x \ \forall \ x \in [-1, 1] \) etc.
\begin{itemize}
  \item $\sin^{-1} x = \csc^{-1} \left( \frac{1}{x} \right)$ $\forall x \in [-1,1]$
  $\tan^{-1} x = \cot^{-1} \left( \frac{1}{x} \right)$ $\forall x > 0$
  $\sec^{-1} x = \cos^{-1} \left( \frac{1}{x} \right)$ $\forall |x| \geq 1$
  \item $\sin^{-1} (-x) = -\sin^{-1} x$ $\forall x \in [-1,1]$
  $\tan^{-1} (-x) = -\tan^{-1} x$ $\forall x \in \mathbb{R}$
  $\cosec^{-1} (-x) = -\cosec^{-1} x$ $\forall |x| \geq 1$
  \item $\cos^{-1} (-x) = \pi - \cos^{-1} x$ $\forall x \in [-1,1]$
  $\cot^{-1} (-x) = \pi - \cot^{-1} x$ $\forall x \in \mathbb{R}$
  $\sec^{-1} (-x) = \pi - \sec^{-1} x$ $\forall |x| \geq 1$

  \item $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $\forall x \in [-1,1]$
  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ $\forall x \in \mathbb{R}$
  $\sec^{-1} x + \cosec^{-1} x = \frac{\pi}{2}$ $\forall |x| \geq 1$

  \item $\tan^{-1} x + \tan^{-1} y = \begin{cases} 
  \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy < 1 \\
  \pi + \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy > 1; x > 0 \\
  -\pi + \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy > 1; x < 0 \\
  \end{cases}$

  \item $\tan^{-1} x - \tan^{-1} y = \begin{cases} 
  \tan^{-1} \frac{x-y}{1+xy} & \text{if } xy < -1 \\
  \pi + \tan^{-1} \frac{x-y}{1+xy} & \text{if } xy < -1; x > 0 \\
  -\pi + \tan^{-1} \frac{x-y}{1+xy} & \text{if } xy < -1; x < 0 \\
  \end{cases}$
\end{itemize}
\[ 2 \tan^{-1} x = \tan^{-1} \left( \frac{2x}{1 - x^2} \right), \quad |x| < 1 \]

\[ 2 \tan^{-1} x = \sin^{-1} \left( \frac{2x}{1 + x^2} \right), \quad |x| \leq 1, \]

\[ 2 \tan^{-1} x = \cos^{-1} \left( \frac{1 - x^2}{1 + x^2} \right), \quad x \geq 0. \]

\[ \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} (\sqrt{1 - y^2} \pm y \sqrt{1 - x^2}) \]

**VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)**

1. Write the principal value of

(i) \( \sin^{-1} \left( -\sqrt{3}/2 \right) \)

(ii) \( \cos^{-1} \left( \sqrt{3}/2 \right) \).

(iii) \( \tan^{-1} \left( -\frac{1}{\sqrt{3}} \right) \)

(iv) \( \cosec^{-1} (-2) \).

(v) \( \cot^{-1} \left( \frac{1}{\sqrt{3}} \right) \).

(vi) \( \sec^{-1} (-2) \).

(vii) \( \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) + \cos^{-1} \left( -\frac{1}{2} \right) + \tan^{-1} \left( -\frac{1}{\sqrt{3}} \right) \)

2. What is the value of the following functions (using principal value).

(i) \( \tan^{-1} \left( \frac{1}{\sqrt{3}} \right) \) - \( \sec^{-1} \left( \frac{2}{\sqrt{3}} \right) \).

(ii) \( \sin^{-1} \left( -\frac{1}{2} \right) - \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \).

(iii) \( \tan^{-1} (1) - \cot^{-1} (-1) \).

(iv) \( \cosec^{-1} (\sqrt{2}) + \sec^{-1} (\sqrt{2}) \).

(v) \( \tan^{-1} (1) + \cot^{-1} (1) + \sin^{-1} (1) \).

(vi) \( \sin^{-1} \left( \sin \frac{4\pi}{5} \right) \).

(vii) \( \tan^{-1} \left( \tan \frac{5\pi}{6} \right) \).

(viii) \( \cosec^{-1} (\cosec \frac{3\pi}{4}) \).
SHORT ANSWER TYPE QUESTIONS (4 MARKS)

3. Show that \( \tan^{-1}\left(\frac{\sqrt{1 + \cos x} + \sqrt{1 - \cos x}}{\sqrt{1 + \cos x} - \sqrt{1 - \cos x}}\right) = \frac{\pi}{4} + \frac{x}{2} \), \( x \in [0, \pi] \)

4. Prove that
\[
\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right) - \cot^{-1}\left(\frac{1 + \cos x}{\sqrt{1 - \cos x}}\right) = \frac{\pi}{4} \quad \text{for} \quad x \in (0, \pi/2).
\]

5. Prove that \( \tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) = \sin^{-1}\frac{x}{a} = \cos^{-1}\left(\frac{\sqrt{a^2 - x^2}}{a}\right) \).

6. Prove that
\[
\cot^{-1}\left[2\tan\left(\cos^{-1}\frac{8}{17}\right)\right] + \tan^{-1}\left[2\tan\left(\sin^{-1}\frac{8}{17}\right)\right] = \tan^{-1}\left(\frac{300}{161}\right).
\]

7. Prove that \( \tan^{-1}\left(\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}\right) = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2 \).

8. Solve: \( \cot^{-1} 2x + \cot^{-1} 3x = \frac{\pi}{4} \).

9. Prove that \( \tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}, m, n > 0 \)

10. Prove that \( \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1 + x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right] = \frac{x+y}{1-xy} \)

11. Solve for \( x, \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right) + \tan^{-1}\left(\frac{-2x}{1-x^2}\right) = \frac{2\pi}{3} \)

12. Prove that \( \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4} \)

13. Solve for \( x, \tan(\cos^{-1} x) = \sin(\tan^{-1} 2); x > 0 \)
14. Prove that \(2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}\left(\frac{32}{43}\right)\)

15. Evaluate \(\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)\right]\)

16. Prove that \(\tan^{-1}\left(\frac{a\cos x - b\sin x}{bc\cos x + as\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x\)

17. Prove that \(\cot\left(\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right) + \cos^{-1} (1 - 2x^2) + \cos^{-1} (2x^2 - 1) = \pi, \ x > 0\)

18. Prove that \(\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right) = 0\) where \(a, b, c > 0\)

19. Solve for \(x\), \(2\tan^{-1}(\cos x) = \tan^{-1}(2\cosec x)\)

20. Express \(\sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)\) in simplest form.

21. If \(\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi\), then prove that \(a + b + c = abc\)

22. If \(\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi\), prove that \(x^2 + y^2 + z^2 + 2xyz = 1\)

   [Hint : Let \(\cos^{-1} x = A, \cos^{-1} y = B, \cos^{-1} z = c\) then \(A + B + C = \pi\)

   or \(A + B = \pi - c\)

   Take \(\cos\) on both the sides].

23. If \(\tan^{-1}\left(\frac{1}{1+1.2}\right) + \tan^{-1}\left(\frac{1}{1+2.3}\right) + \ldots + \tan^{-1}\left(\frac{1}{1+n \cdot (n+1)}\right) = \tan^{-1} \theta\) then find the value of \(\theta\).

24. If \((\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}\) then find \(x\).

25. If \(\sin [\cot^{-1} (x + 1)] = \cos (\tan^{-1} x)\), then find \(x\).
ANSWERS

1. (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{6}$ (iii) $-\frac{\pi}{6}$ (iv) $-\frac{\pi}{6}$
   (v) $\frac{\pi}{3}$ (vi) $\frac{2\pi}{3}$ (vii) $\frac{\pi}{6}$.

2. (i) 0 (ii) $-\frac{\pi}{3}$ (iii) $-\frac{\pi}{2}$ (iv) $\frac{\pi}{2}$
   (v) $\pi$ (vi) $\frac{\pi}{5}$ (vii) $-\frac{\pi}{6}$ (viii) $\frac{\pi}{4}$.

8. 1

11. $\tan \frac{\pi}{12} = 2 - \sqrt{3}$

13. $\frac{\sqrt{5}}{3}$

15. $\frac{\sqrt{11}-3}{\sqrt{2}}$

19. $x = \frac{\pi}{4}$.

20. $\sin^{-1} x - \sin^{-1} \sqrt{x}$.

21. Hint: Let $\tan^{-1} a = \alpha$
   $\tan^{-1} b = \beta$
   $\tan^{-1} c = \gamma$

   then given, $\alpha + \beta + \gamma = \pi$

   $\therefore \alpha + \beta = \pi - \gamma$

   take tangent on both sides,
   $\tan (\alpha + \beta) = \tan (\pi - \gamma)$

23. $\theta = \frac{n}{n+2}$

24. $x = -1$

25. $x = -\frac{1}{2}$
CHAPTER 3 & 4

MATRICES AND DETERMINANTS

POINTS TO REMEMBER

- **Matrix**: A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements of the matrix.

- Numbers written in the horizontal line form a row of the matrix. Numbers written in the vertical line form a column of the matrix.

- **Order of Matrix**: A matrix having ‘m’ rows and ‘n’ columns is called the matrix of order \( m \times n \).

- **Square Matrix**: An \( m \times n \) matrix is said to be a square matrix if \( m = n \).

- **Column Matrix**: A matrix having only one column is called a column matrix i.e. \( A = [a_{ij}]_{mx1} \) is a column matrix of order \( mx1 \).

- **Row Matrix**: A matrix having only one row is called a row matrix i.e. \( B = [b_{ij}]_{1xn} \) is a row matrix of order \( 1xn \).

- **Zero Matrix**: A matrix having all the elements zero is called zero matrix or null matrix.

- **Diagonal Matrix**: A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

- **Scalar Matrix**: A diagonal matrix in which all diagonal elements are equal is called a scalar matrix.

- **Identity Matrix**: A scalar matrix in which each diagonal element is 1, is called an identity matrix or a unit matrix. It is denoted by \( I \).

\[
I = [e_{ij}]_{n \times n}
\]

where, \( e_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases} \)
Transpose of a Matrix: If \( A = [a_{ij}]_{m \times n} \) be an \( m \times n \) matrix then the matrix obtained by interchanging the rows and columns of \( A \) is called the transpose of the matrix. Transpose of \( A \) is denoted by \( A' \) or \( A^T = [a_{ji}]_{n \times m} \).

Properties of the transpose of a matrix.

(i) \( (A')' = A \)  
(ii) \( (A + B)' = A' + B' \)  
(iii) \( (kA)' = kA' \), \( k \) is a scalar  
(iv) \( (AB)' = B'A' \)

Symmetric Matrix: A square matrix \( A = [a_{ij}] \) is symmetric if \( a_{ij} = a_{ji} \) \( \forall \ i, j \). Also a square matrix \( A \) is symmetric if \( A' = A \).

Skew Symmetric Matrix: A square matrix \( A = [a_{ij}] \) is skew-symmetric, if \( a_{ij} = -a_{ji} \) \( \forall \ i, j \). Also a square matrix \( A \) is skew-symmetric, if \( A' = -A \).

Determinant: To every square matrix \( A = [a_{ij}] \) of order \( n \times n \), we can associate a number (real or complex) called determinant of \( A \). It is denoted by \( \det A \) or \( |A| \).

Properties

(i) \( |AB| = |A| \cdot |B| \), where \( A \) & \( B \) are square matrices.

(ii) \( |kA|_{n \times n} = k^n \cdot |A|_{n \times n} \) where \( k \) is a scalar.

(iii) Area of triangle with vertices \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) is given by

\[
\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}
\]

The points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) are collinear \( \iff \) \[
\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0
\]

Adjoint of a Square Matrix \( A \) is the transpose of the matrix whose elements have been replaced by their cofactors and is denoted as \( adj \ A \).

Let \( A = [a_{ij}]_{m \times n} \)

\( adj \ A = [A_{ji}]_{m \times n} \)

where \( A_{ji} \) is cofactor of \( a_{ij} \).
Properties

(i) \( A(adj \ A) = (adj \ A) \ A = |A| \ I \)

(ii) If \( A \) is a square matrix of order \( n \) then \( |adj \ A| = |A|^{n-1} \)

(iii) \( adj \ (AB) = (adj \ B) \ (adj \ A) \).

(iv) \( |k \ adj \ A| = k^n |A|^{n-1} \).

Note: Correctness of \( adj \ A \) can be checked by using \( A.(adj \ A) = (adj \ A) \cdot A = |A| \ I \)

Singular Matrix: A square matrix is called singular if \( |A| = 0 \), otherwise it will be called a non-singular matrix.

Inverse of a Matrix: A square matrix whose inverse exists, is called invertible matrix. Inverse of only a non-singular matrix exists. Inverse of a matrix \( A \) is denoted by \( A^{-1} \) and is given by

\[
A^{-1} = \frac{1}{|A|} \cdot adj \ A
\]

Properties

(i) \( AA^{-1} = A^{-1}A = I \)

(ii) \( (A^{-1})^{-1} = A \)

(iii) \( (AB)^{-1} = B^{-1}A^{-1} \)

(iv) \( (A^T)^{-1} = (A^{-1})^T \)

(v) \( |A^{-1}| = \frac{1}{|A|}, \ |A| \neq 0 \)

- Solution of system of equations using matrices:

If \( AX = B \) is a matrix equation then its solution is \( X = A^{-1}B \).

(i) If \( |A| \neq 0 \), system is consistent and has a unique solution.

(ii) If \( |A| = 0 \) and \( (adj \ A) B \neq 0 \) then system is inconsistent and has no solution.

(iii) If \( |A| = 0 \) and \( (adj \ A) B = 0 \) then system is either consistent and has infinitely many solutions or system is inconsistent and has no solution.
VERY SHORT ANSWER TYPE QUESTIONS (1 Mark)

1. If \[
\begin{bmatrix}
x+3 & 4 \\
y-4 & x+y
\end{bmatrix}
= \begin{bmatrix}
5 & 4 \\
3 & 9
\end{bmatrix}
\], find \( x \) and \( y \).

2. If \( A = \begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix} \) and \( B = \begin{bmatrix}
0 & i \\
i & 0
\end{bmatrix} \), find \( AB \).

3. Find the value of \( a_{23} + a_{32} \) in the matrix \( A = [a_{ij}]_{3 \times 3} \)

where \( a_{ij} = \begin{cases}
|2i - j| & \text{if } i > j \\
-i + 2j + 3 & \text{if } i \leq j
\end{cases} \).

4. If \( B \) be a \( 4 \times 5 \) type matrix, then what is the number of elements in the third column.

5. If \( A = \begin{bmatrix}
5 & 2 \\
0 & 9
\end{bmatrix} \) and \( B = \begin{bmatrix}
3 & 6 \\
0 & -1
\end{bmatrix} \), find \( 3A - 2B \).

6. If \( A = \begin{bmatrix}
2 & -3 \\
-7 & 5
\end{bmatrix} \) and \( B = \begin{bmatrix}
1 & 0 \\
2 & -6
\end{bmatrix} \), find \( (A+B)' \).

7. If \( A = \begin{bmatrix}
1 & 0 & 4
\end{bmatrix} \) and \( B = \begin{bmatrix}
2 & 5 \\
6
\end{bmatrix} \), find \( AB \).

8. If \( A = \begin{bmatrix}
4 & x+2 \\
2x-3 & x+1
\end{bmatrix} \) is symmetric matrix, then find \( x \).

9. For what value of \( x \) the matrix \( \begin{bmatrix}
0 & 2 & -3 \\
-2 & 0 & -4 \\
3 & 4 & x+5
\end{bmatrix} \) is skew symmetric matrix.

10. If \( A = \begin{bmatrix}
2 & 3 \\
1 & 0
\end{bmatrix} = P + Q \) where \( P \) is symmetric and \( Q \) is skew-symmetric matrix, then find the matrix \( Q \).

11. Find the value of \[
\begin{bmatrix}
a + ib & c + id \\
-c + id & a - ib
\end{bmatrix}
\]
12. If \[
\begin{vmatrix}
2x + 5 & 3 \\
5x + 2 & 9
\end{vmatrix} = 0,
\]
find \(x\).

13. For what value of \(k\), the matrix \[
\begin{bmatrix}
k & 2 \\
3 & 4
\end{bmatrix}
\]
has no inverse.

14. If \(A = \begin{bmatrix}
\sin 30^\circ & \cos 30^\circ \\
-\sin 60^\circ & \cos 60^\circ
\end{bmatrix}\), what is \(|A|\).

15. Find the cofactor of \(a_{12}\) in \[
\begin{vmatrix}
2 & -3 & 5 \\
6 & 0 & 4 \\
1 & 5 & -7
\end{vmatrix}.
\]

16. Find the minor of \(a_{23}\) in \[
\begin{vmatrix}
1 & 3 & -2 \\
4 & -5 & 6 \\
3 & 5 & 2
\end{vmatrix}.
\]

17. Find the value of \(P\), such that the matrix \[
\begin{bmatrix}
-1 & 2 \\
4 & P
\end{bmatrix}
\]
is singular.

18. Find the value of \(x\) such that the points \((0, 2)\), \((1, x)\) and \((3, 1)\) are collinear.

19. Area of a triangle with vertices \((k, 0)\), \((1, 1)\) and \((0, 3)\) is 5 unit. Find the value \((s)\) of \(k\).

20. If \(A\) is a square matrix of order 3 and \(|A| = -2\), find the value of \(|-3A|\).

21. If \(A = 2B\) where \(A\) and \(B\) are square matrices of order \(3 \times 3\) and \(|B| = 5\), what is \(|A|\)?

22. (i) What is the number of all possible matrices of order \(2 \times 3\) with each entry 0, 1 or 2.

(ii) What is the number of all possible non zero matrices of order \(2 \times 3\) with each entry 0, 1 or 2.

23. Find the area of the triangle with vertices \((0, 0)\), \((6, 0)\) and \((4, 3)\).

24. If \[
\begin{vmatrix}
2x & 4 \\
-1 & x
\end{vmatrix} = \begin{vmatrix}
6 & -3 \\
2 & 1
\end{vmatrix},
\]
find \(x\).

25. If \(A = \begin{bmatrix}
x + y & y + z & z + x \\
z & x & y \\
1 & 1 & 1
\end{bmatrix}\), write the value of \(\det A\).
26. Write the value of the following determinant
\[
\begin{vmatrix}
2 & 3 & 4 \\
5 & 6 & 8 \\
6x & 9x & 12x
\end{vmatrix}
\]
27. If \( A \) is a non-singular matrix of order 3 and \( |A| = -3 \) find \( |adj\ A| \).
28. If \( A = \begin{bmatrix} 5 & -3 \\ 6 & 8 \end{bmatrix} \) find \( (adj\ A) \).
29. Given a square matrix \( A \) of order 3 \( \times \) 3 such that \( |A| = 12 \) find the value of \( |A\ adj\ A| \).
30. If \( A \) is a square matrix of order 3 such that \( |adj\ A| = 81 \) find \( |A| \).
31. Let \( A \) be a non-singular square matrix of order 3 \( \times \) 3 find \( |adj\ A| \) if \( |A| = 10 \).
32. If \( A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \) find \( |(A^{-1})^{-1}| \).
33. If \( A = [-1 \ 2 \ 3] \) and \( B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} \) find \( |AB| \).

**SHORT ANSWER TYPE QUESTIONS (4 MARKS)**

34. Find \( x, y, z \) and \( w \) if \( \begin{bmatrix} x - y & 2x + z \\ 2x - y & 3x + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix} \).
35. Construct a 3 \( \times \) 3 matrix \( A = [a_{ij}] \) whose elements are given by
\[
a_{ij} = \begin{cases} 
1 + i + j & \text{if } i \geq j \\
|i - 2j| & \text{if } i < j
\end{cases}
\]
36. Find \( A \) and \( B \) if \( 2A + 3B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix} \) and \( A - 2B = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 6 & 2 \end{bmatrix} \).
37. If \( A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \) and \( B = [-2 \ -1 \ -4] \), verify that \( (AB)^\prime = B^\prime A^\prime \).
38. Express the matrix
\[
\begin{bmatrix}
3 & 3 & -1 \\
-2 & -2 & 1 \\
-4 & -5 & 2
\end{bmatrix}
= P + Q
\] where \( P \) is a symmetric and \( Q \) a skew-symmetric matrix.

39. If \( A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \), then prove that \( A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix} \) where \( n \) is a natural number.

40. Let \( A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \), \( B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} \), \( C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \), find a matrix \( D \) such that \( CD - AB = O \).

41. Find the value of \( x \) such that
\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 5 & 1 \\
15 & 3 & 2
\end{bmatrix}
x = \begin{bmatrix}
1 \\
2 \\
x
\end{bmatrix}
\]

42. Prove that the product of the matrices
\[
\begin{bmatrix}
\cos^2 \theta & \cos \theta \sin \theta \\
\cos \theta \sin \theta & \sin^2 \theta
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\cos^2 \phi & \cos \phi \sin \phi \\
\cos \phi \sin \phi & \sin^2 \phi
\end{bmatrix}
\]
is the null matrix, when \( \theta \) and \( \phi \) differ by an odd multiple of \( \frac{\pi}{2} \).

43. If \( A = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix} \), show that \( A^2 - 12A - I = 0 \). Hence find \( A^{-1} \).

44. Show that \( A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \) satisfies the equation \( x^2 - 6x + 17 = 0 \). Hence find \( A^{-1} \).

45. If \( A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \), find \( x \) and \( y \) such that \( A^2 - xA + yI = 0 \).

46. Find the matrix \( X \) so that \( X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix} \).
47. If \( A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \) then show that \( (AB)^{-1} = B^{-1}A^{-1} \).

48. Test the consistency of the following system of equations by matrix method:

\[
3x - y = 5; \quad 6x - 2y = 3
\]

49. Using elementary row transformations, find the inverse of the matrix

\[
A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}, \text{if possible.}
\]

50. By using elementary column transformations, find the inverse of

(i) \( A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \)

(ii) \( A = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix} \)

(iii) \( A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \)

51. If \( A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \) and \( A + A' = I \), then find the general value of \( \alpha \).

Using properties of determinants, prove the following: Q 52 to Q 59.

52. \[
\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3
\]

53. \[
x + 2 \quad x + 3 \quad x + 2a \\
x + 3 \quad x + 4 \quad x + 2b = 0 \text{ if } a, b, c \text{ are in } AP.
\]

54. \[
\begin{vmatrix} \sin \alpha & \cos \alpha & \sin(\alpha + \delta) \\ \sin \beta & \cos \beta & \sin(\beta + \delta) \\ \sin \gamma & \cos \gamma & \sin(\gamma + \delta) \end{vmatrix} = 0
\]

55. \[
\begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2.
\]

[Class XII : Maths] [30]
56. \[
\begin{bmatrix}
  a + b & c + a & a + b \\
  p + q & r + p & p + q \\
  x + y & z + x & x + y
\end{bmatrix} = \begin{bmatrix}
  a & b & c \\
  p & q & r \\
  x & y & z
\end{bmatrix}.
\]

57. \[
\begin{vmatrix}
  a^2 & bc & ac + c^2 \\
  a^2 + ab & b^2 & ac \\
  ab & b^2 + bc & c^2
\end{vmatrix} = 4a^2b^2c^2.
\]

58. \[
\begin{vmatrix}
  x + a & b & c \\
  a & x + b & c \\
  a & b & x + c
\end{vmatrix} = x^2(x + a + b + c).
\]

59. Show that:
\[
\begin{vmatrix}
  x & y & z \\
  x^2 & y^2 & z^2 \\
  yz & zx & xy
\end{vmatrix} = (y - z)(z - x)(x - y)(yz + zx + xy).
\]

60. (i) If the points \((a, b)\) \((a', b')\) and \((a - a', b - b')\) are collinear, show that \(ab' = a'b\).

(ii) If \(A = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\) and \(B = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix}\) verify that \(|AB| = |A||B|\).

61. Given \(A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\) and \(B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\). Find the product \(AB\) and also find \((AB)^{-1}\).

62. Solve the following equation for \(x\).
\[
\begin{vmatrix}
  a + x & a - x & a - x \\
  a - x & a + x & a - x \\
  a - x & a - x & a + x
\end{vmatrix} = 0.
\]

63. If \(A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}\) and \(I\) is the identity matrix of order 2, show that,
\[
I + A = (I - A) \begin{bmatrix}
  \cos \alpha & -\sin \alpha \\
  \sin \alpha & \cos \alpha
\end{bmatrix}.
\]
64. Use matrix method to solve the following system of equations:
   \[\begin{align*}
   5x - 7y &= 2, \\
   7x - 5y &= 3.
   \end{align*}\]

65. If \(A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\) and \(I\) is the identity matrix of order 2, then show that
   \(A^2 = 4A - 3I\) and hence find \(A^{-1}\).

66. If \(A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\) and \(B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\) and \((A + B)^2 = A^2 + B^2\), then find the values of \(a\) and \(b\).

67. Using properties of determinants, prove that
   \[
   \begin{vmatrix}
   1 & a & a^2 \\
   a^2 & 1 & a \\
   a & a^2 & 1
   \end{vmatrix}
   = (1 - a^3)^2
   \]

68. In a parliament election, a political party hired a public relations firm to promote its candidates in three ways—telephone, house calls and letters. The cost per contact (in paise) is given in matrix \(A\) as
   \[
   A = \begin{bmatrix}
   140 & \text{telephone} \\
   200 & \text{house calls} \\
   150 & \text{letters}
   \end{bmatrix}
   \]

The number of contacts of each type made in two cities \(X\) and \(Y\) is given in the matrix \(B\) as
   \[
   B = \begin{bmatrix}
   1000 & 500 & 5000 & \text{City X} \\
   3000 & 1000 & 10000 & \text{City Y}
   \end{bmatrix}
   \]

Find the total amount spent by the party in the two cities. What should one consider before casting his/her vote—Party’s promotional activity or their social activities?

**LONG ANSWER TYPE QUESTIONS (6 MARKS)**

69. Obtain the inverse of the following matrix using elementary operations
   \[
   (i) \begin{bmatrix}
   2 & -1 & 4 \\
   4 & 0 & 2 \\
   3 & -2 & 7
   \end{bmatrix}
   \text{ and } (ii) \begin{bmatrix}
   1 & 2 & 3 \\
   2 & 5 & 7 \\
   -2 & -4 & -5
   \end{bmatrix}
   \]
70. If \( A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \) are two square matrices, find \( AB \) and hence solve the system of linear equations:
\[
\begin{align*}
x - y & = 3, \\
2x + 3y + 4z & = 17, \\
y + 2z & = 7.
\end{align*}
\]

71. Solve the following system of equations by matrix method, where \( x \neq 0, y \neq 0, z \neq 0 \)
\[
\begin{align*}
\frac{2}{x} - \frac{3}{y} + \frac{3}{z} & = 10, \\
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} & = 10, \\
\frac{3}{x} - \frac{1}{y} + \frac{2}{z} & = 13.
\end{align*}
\]

72. Find \( A^{-1} \), where \( A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix} \), hence solve the system of linear equations:
\[
\begin{align*}
x + 2y - 3z & = -4, \\
2x + 3y + 2z & = 2, \\
3x - 3y - 4z & = 11
\end{align*}
\]

73. The sum of three numbers is 2. If we subtract the second number from twice the first number, we get 3. By adding double the second number and the third number we get 0. Represent it algebraically and find the numbers using matrix method.

74. Compute the inverse of the matrix.
\[
A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 5 \end{bmatrix}
\]
and verify that \( A^{-1} A = I_3 \).

75. If the matrix \( A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \) and \( B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \), then compute \((AB)^{-1}\).

76. Using matrix method, solve the following system of linear equations:
\[
\begin{align*}
2x - y & = 4, \\
2y + z & = 5, \\
z + 2x & = 7.
\end{align*}
\]
77. Find \( A^{-1} \) if \( A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \). Also show that \( A^{-1} = \frac{A^2 - 3I}{2} \).

78. Show that
\[
\begin{vmatrix} b^2 + c^2 & ab & ac \\ ba & c^2 + a^2 & bc \\ ca & cb & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2
\]

79. Show that
\[
\begin{vmatrix} a & b - c & c + b \\ a + c & b & c - a \\ a - b & b + a & c \end{vmatrix} = (a + b + c)(a^2 + b^2 + c^2)
\]

80. If \( A = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \), verify that \( A \cdot (adj \ A) = (adj \ A) \cdot A = |A| I_3 \).

81. For the matrix \( A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \), verify that \( A^3 - 6A^2 + 9A - 4I = 0 \), hence find \( A^{-1} \).

82. Find the matrix \( X \) for which
\[
\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} X \cdot \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}
\]

83. By using properties of determinants prove the following :
\[
\begin{vmatrix} 1 + a^2 - b^2 & 2ab & -2b \\ 2ab & 1 - a^2 + b^2 & 2a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix} = (1 + a^2 + b^2)^3.
\]

84. \[
\begin{vmatrix} (y + z)^2 & xy & zy \\ xy & (x + z)^2 & zx \\ xz & yz & (x + y)^2 \end{vmatrix} = 2xyz(x + y + z)^3.
\]
85. \[
\begin{vmatrix}
a & a+b & a+b+c \\
2a & 3a+2b & 4a+3b+2c \\
3a & 6a+3b & 10a+6b+3c \\
\end{vmatrix} = a^3.
\]

86. If \(x, y, z\) are different and \[
\begin{vmatrix}
x & x^2 & 1+x^3 \\
y & y^2 & 1+y^3 \\
z & z^2 & 1+z^3 \\
\end{vmatrix} = 0,
\]
show that \(xyz = -1\).

87. If \(x, y, z\) are the 10th, 13th and 15th terms of a G.P. find the value of
\[
\Delta = \begin{vmatrix}
\log x & 10 & 1 \\
\log y & 13 & 1 \\
\log z & 15 & 1 \\
\end{vmatrix},
\]

88. Using the properties of determinants, show that:
\[
\begin{vmatrix}
1+a & 1 & 1 \\
1 & 1+b & 1 \\
1 & 1 & 1+c \\
\end{vmatrix} = abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab
\]

89. Using properties of determinants prove that
\[
\begin{vmatrix}
-bc & b^2+bc & c^2+bc \\
a^2+ac & -ac & c^2+ac \\
a^2+ab & b^2+ab & -ab \\
\end{vmatrix} = (ab+bc+ca)^3
\]

90. If \(A = \begin{bmatrix} 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}\), find \(A^{-1}\) and hence solve the system of equations
\[
3x + 4y + 7z = 14, \ 2x - y + 3z = 4, \ x + 2y - 3z = 0.
\]

ANSWERS

1. \(x = 2, \ y = 7\) 
2. \[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}
\]
3. 11. 
4. 4
5. \[
\begin{bmatrix}
9 & -6 \\
0 & 29
\end{bmatrix}.
\]

6. \[
\begin{bmatrix}
3 & -5 \\
-3 & -1
\end{bmatrix}.
\]

7. \(AB = [26].\)

8. \(x = 5\)

9. \(x = -5\)

10. \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]

11. \(a^2 + b^2 + c^2 + d^2.\)

12. \(x = -13\)

13. \(k = \frac{3}{2}\)

14. \(|A| = 1.\)

15. \(46\)

16. \(-4\)

17. \(P = -8\)

18. \(x = \frac{5}{3}\)

19. \(k = \frac{-7}{2}, \frac{13}{2} \)

20. \(54\)

21. \(40\)

22. (i) 729  (ii) 728

23. 9 sq. units

24. \(x = \pm 2\)

25. 0

26. 0

27. \(9\)

28. \[
\begin{bmatrix}
8 & 3 \\
-6 & 5
\end{bmatrix}.
\]

29. 1728

30. \(|A| = \pm 9\)

31. 100

32. 11

33. \(|AB| = -11\)

34. \(x = 1, y = 2, z = 3, w = 10\)

35. \[
\begin{bmatrix}
3 & 3/2 & 5/2 \\
4 & 5 & 2 \\
5 & 6 & 7
\end{bmatrix}.
\]
36. \[ A = \begin{bmatrix} \frac{11}{7} & \frac{4}{7} & \frac{9}{7} \\ \frac{1}{7} & \frac{4}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{4}{7} & \frac{1}{7} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{12}{7} & \frac{5}{7} \\ \frac{7}{7} & \frac{-7}{7} & \frac{-7}{7} \end{bmatrix} \]

38. \[ \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ \frac{3}{2} & -3 & 0 \end{bmatrix} \]

40. \[ D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix} \]

41. \[ x = -2 \text{ or } -14 \]

43. \[ A^{-1} = \begin{bmatrix} -7 & 3 \\ 12 & -5 \end{bmatrix} \]

44. \[ A^{-1} = \frac{1}{17} \begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix} \]

45. \[ x = 9, \quad y = 14 \]

46. \[ x = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \]

48. Inconsistent

50. (i) \[ A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \] (ii) \[ A^{-1} = \begin{bmatrix} 7 & -1 \\ -10 & 7 \end{bmatrix} \] (iii) \[ A^{-1} = \frac{1}{17} \begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix} \]

51. \[ \alpha = 2n\pi \pm \frac{\pi}{3}, \quad n \in \mathbb{Z} \]

61. \[ AB = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad (AB)^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \]

62. \[ 0, \quad 3a \]

64. \[ x = \frac{11}{24}, \quad y = \frac{1}{24} \]
65. \[ A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

66. \( a = 1, \ b = 4 \)

68. \( 990000, \ 2120000 \)

69. (i) \[ A^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix} \]

(ii) \[ A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \]

70. \( AB = 6I \)

71. \( x = \frac{1}{2}, \ y = \frac{1}{3}, \ z = \frac{1}{5} \)

72. \[ A^{-1} = \frac{1}{67} \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \]

73. \( x = 1, \ y = -2, \ z = 2 \)

74. \[ A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \]

75. \( (AB)^{-1} = \frac{1}{19} \begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix} \)

76. \( x = 3, \ y = 2, \ z = 1 \)

77. \[ A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \]

81. \[ A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \]

82. \[ X = \begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix} \]

87. \( 0 \)

90. \[ A^{-1} = \frac{1}{62} \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix} ; \ x = 1, \ y = 1, \ z = 1. \]
A function \( f(x) \) is said to be continuous at \( x = c \) iff \( \lim_{x \to c} f(x) = f(c) \)

\[ i.e., \quad \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c) \]

- \( f(x) \) is continuous in \((a, b)\) iff it is continuous at \( x = c \forall c \in (a, b) \).

- \( f(x) \) is continuous in \([a, b]\) iff
  
  (i) \( f(x) \) is continuous in \((a, b)\)

  (ii) \( \lim_{x \to a^+} f(x) = f(a) \),

  (iii) \( \lim_{x \to b^-} f(x) = f(b) \)

- Trigonometric functions are continuous in their respective domains.

- Every polynomial function is continuous on \( \mathbb{R} \).

- If \( f(x) \) and \( g(x) \) are two continuous functions at \( x = a \) and \( c \in \mathbb{R} \) then
  
  (i) \( f(x) \pm g(x) \) are also continuous functions at \( x = a \).

  (ii) \( g(x) \cdot f(x), f(x) + c, cf(x), |f(x)| \) are also continuous at \( x = a \).

  (iii) \( \frac{f(x)}{g(x)} \) is continuous at \( x = a \) provided \( g(a) \neq 0 \).

- \( f(x) \) is derivable at \( x = c \) in its domain iff
The value of above limit is denoted by \( f'(c) \) and is called the derivative of \( f(x) \) at \( x = c \).

- \( \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \)

- \( \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \)

- If \( y = f(u) \) and \( u = g(t) \) then \( \frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u) \cdot g'(t) \) (Chain Rule)

- If \( y = f(u) \), \( x = g(u) \) then,
  \[
  \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{f'(u)}{g'(u)}.
  \]

- \( \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \), \( \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}} \)

- \( \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2} \), \( \frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1 + x^2} \)

- \( \frac{d}{dx} (\sec^{-1} x) = \frac{1}{x \sqrt{x^2 - 1}} \), \( \frac{d}{dx} (\cosec^{-1} x) = -\frac{1}{x \sqrt{x^2 - 1}} \)

- \( \frac{d}{dx} (e^x) = e^x \), \( \frac{d}{dx} (\log x) = \frac{1}{x} \)

- \( f(x) = \lfloor x \rfloor \) is discontinuous at all integral points and continuous for all \( x \in \mathbb{R} - \mathbb{Z} \).

- Rolle’s theorem: If \( f(x) \) is continuous in \([a, b]\), derivable in \((a, b)\) and \( f(a) = f(b) \) then there exists atleast one real number \( c \in (a, b) \) such that \( f'(c) = 0 \).
Mean Value Theorem: If \( f(x) \) is continuous in \([a, b]\) and derivable in \((a, b)\) then there exists at least one real number \( c \in (a, b) \) such that
\[
 f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

\( f(x) = \log_e x, \; (x > 0) \) is continuous function.

**VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)**

1. For what value of \( x \), \( f(x) = |2x - 7| \) is not derivable.
2. Write the set of points of continuity of \( g(x) = |x - 1| + |x + 1| \).
3. What is derivative of \(|x - 3|\) at \( x = -1 \).
4. What are the points of discontinuity of \( f(x) = \frac{(x - 1) + (x + 1)}{(x - 7)(x - 6)} \).
5. Write the number of points of discontinuity of \( f(x) = [x] \) in \([3, 7]\).
6. The function, \( f(x) = \begin{cases} \lambda x - 3 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases} \) is a continuous function for all \( x \in \mathbb{R} \), find \( \lambda \).
7. For what value of \( K \), \( f(x) = \begin{cases} \tan 3x, & x \neq 0 \\ \sin 2x, & x = 0 \end{cases} \) is continuous \( \forall x \in \mathbb{R} \).
8. Write derivative of \( \sin x \) w.r.t. \( \cos x \).
9. If \( f(x) = x^2 g(x) \) and \( g(1) = 6 \), \( g'(x) = 3 \) find value of \( f'(1) \).
10. Write the derivative of the following functions:
   
   (i) \( \log_3 (3x + 5) \)  
   (ii) \( e^{\log_2 x} \)  
   (iii) \( e^{6 \log_e (x-1)}, \; x > 1 \)
(iv) \( \sec^{-1}\sqrt{x} + \csc^{-1}\sqrt{x}, x \geq 1 \).

(v) \( \sin^{-1}(x^{7/2}) \)

(vi) \( \log_x 5, x > 0 \).

**SHORT ANSWER TYPE QUESTIONS (4 MARKS)**

11. Discuss the continuity of following functions at the indicated points.

(i) \( f(x) = \begin{cases} \frac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases} \) at \( x = 0 \).

(ii) \( g(x) = \begin{cases} \frac{\sin 2x}{3x}, & x \neq 0 \\ \frac{3}{2}, & x = 0 \end{cases} \) at \( x = 0 \).

(iii) \( f(x) = \begin{cases} x^2 \cos \left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \) at \( x = 0 \).

(iv) \( f(x) = |x| + |x - 1| \) at \( x = 1 \).

(v) \( f(x) = \begin{cases} x - [x], & x \neq 1 \\ 0, & x = 1 \end{cases} \) at \( x = 1 \).

12. For what value of \( k \), \( f(x) = \begin{cases} 3x^2 - kx + 5, & 0 \leq x < 2 \\ 1 - 3x, & 2 \leq x \leq 3 \end{cases} \) is continuous \( \forall x \in [0,3] \).

13. For what values of \( a \) and \( b \)

\[ f(x) = \begin{cases} \frac{x + 2}{|x + 2|} + a & \text{if } x < -2 \\ a + b & \text{if } x = -2 \\ \frac{x + 2}{|x + 2|} + 2b & \text{if } x > -2 \end{cases} \] is continuous at \( x = 2 \).
14. Prove that \( f(x) = |x + 1| \) is continuous at \( x = -1 \), but not derivable at \( x = -1 \).

15. For what value of \( p \),
\[
    f(x) = \begin{cases} 
        x^p \sin(1/x) & x \neq 0 \\
        0 & x = 0 
    \end{cases}
\]
is derivable at \( x = 0 \).

16. If \( y = \frac{1}{2} \left[ \tan^{-1} \left( \frac{2x}{1 - x^2} \right) + 2 \tan^{-1} \left( \frac{1}{x} \right) \right] \), \( 0 < x < 1 \), find \( \frac{dy}{dx} \).

17. If \( y = \sin \left[ 2 \tan^{-1} \left( \frac{1 - x}{1 + x} \right) \right] \) then \( \frac{dy}{dx} = ? \)

18. If \( 5^x + 5^y = 5^{x+y} \) then prove that \( \frac{dy}{dx} + 5^{y-x} = 0 \).

19. If \( x \sqrt{1-y^2} + y \sqrt{1-x^2} = a \) then show that \( \frac{dy}{dx} = -\frac{1-y^2}{\sqrt{1-x^2}} \).

20. If \( \sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y) \) then show that \( \frac{dy}{dx} = \frac{1-y^2}{\sqrt{1-x^2}} \).

21. If \( (x + y)^m + n = x^m \cdot y^n \) then prove that \( \frac{dy}{dx} = \frac{y}{x} \).

22. Find the derivative of \( \tan^{-1} \left( \frac{2x}{1-x^2} \right) \) w.r.t. \( \sin^{-1} \left( \frac{2x}{1+x^2} \right) \).

23. Find the derivative of \( \log_e(\sin x) \) w.r.t. \( \log_a(\cos x) \).

24. If \( x^y + y^x + x^x = m^n \), then find the value of \( \frac{dy}{dx} \).

25. If \( x = a \cos^3 \theta, \ y = a \sin^3 \theta \) then find \( \frac{d^2 y}{dx^2} \).
26. If \( x = ae^t \) (sint – cost) then show that \( \frac{dy}{dx} \) at \( t = \frac{\pi}{4} \) is 1.

27. If \( y = \sin^{-1} \left[ x \sqrt{1 - x^2} - \sqrt{x \sqrt{1 - x^2}} \right] \) then find \( \frac{dy}{dx} \).

28. If \( y = x^{\log_e x} + (\log_e x)^x \) then find \( \frac{dy}{dx} \).

29. Differentiate \( x^{x^x} \) w.r.t. \( x \).

30. Find \( \frac{dy}{dx} \), if \( (\cos x)^y = (\cos y)^x \).

31. If \( y = \tan^{-1} \left( \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}} \right) \) where \( \frac{\pi}{2} < x < \pi \) find \( \frac{dy}{dx} \).

Hint: \( \sin \frac{x}{2} > \cos \frac{x}{2} \) for \( x \in \left( \frac{\pi}{2}, \pi \right) \).

32. If \( x = \sin \left( \frac{1}{a} \log_e y \right) \) then show that \( (1 - x^2) \frac{d^2 y}{dx^2} - xy' - a^2y = 0 \).

33. Differentiate \( (\log x)^{\log x}, x > 1 \) w.r.t. \( x \).

34. If \( \sin y = x \sin (a + y) \) then show that \( \frac{dy}{dx} = \frac{\sin^2 (a + y)}{\sin a} \).

35. If \( y = \sin^{-1} x \), find \( \frac{d^2 y}{dx^2} \) in terms of \( y \).

36. If \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), then show that \( \frac{d^2 y}{dx^2} = \frac{-b^4}{a^2 y^3} \).

37. If \( y = e^{a \cos^{-1} x}, -1 \leq x \leq 1 \) show that \( (1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0 \).

38. If \( y^3 = 3ax^2 - x^3 \) then prove that \( \frac{d^2 y}{dx^2} = \frac{-2a^2 x^2}{y^5} \).
39. Verify Rolle’s theorem for the function, \( y = x^2 + 2 \) in the interval \([a, b]\) where \( a = -2, \ b = 2 \).

40. Verify Mean Value Theorem for the function, \( f(x) = x^2 \) in \([2, 4]\)

**ANSWERS**

1. \( x = \frac{7}{2} \).

2. \( R \)

3. \(-1\)

4. \( x = 6, 7 \)

5. Points of discontinuity of \( f(x) \) are 4, 5, 6, 7 i.e. four points.

   *Note*: At \( x = 3 \), \( f(x) = [x] \) is continuous. because \( \lim_{{x \to 3}} f(x) = 3 = f(3) \).

6. \( \lambda = \frac{7}{2} \).

7. \( \kappa = \frac{3}{4} \).

8. \(-\cot x\)

9. 15

10. (i) \( \frac{3}{3x + 5} \log_3 e \)

    (ii) \( e^{\log_2 x} \cdot \frac{1}{x} \log_2 e \).

    (iii) \( 6 \ (x - 1)^5 \)

    (iv) 0

    (v) \( \frac{7 \ x^2 \sqrt{x}}{2 \sqrt{1 - x^2}} \).

    (vi) \( \frac{-\log_3 5}{x \ (\log_3 x)^2} \).

11. (i) Discontinuous

    (ii) Discontinuous

    (iii) Continuous

    (iv) continuous

    (v) Discontinuous

12. \( k = 11 \)

13. \( a = 0, \ b = -1 \).

15. \( p > 1 \).

16. 0

17. \( \frac{-x}{\sqrt{1 - x^2}} \)

22. 1

23. \(-\cot^2 x \log_3 a\)
24. \[ \frac{dy}{dx} = -x^x (1 + \log x) - yx^{y-1} - y^x \log y \\]
\[ x^y \log x + xy^{x-1} \]

25. \[ \frac{d^2 y}{dx^2} = \frac{1}{3a} \csc \theta \sec^4 \theta. \]

27. \[ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{2\sqrt{x} \sqrt{1 - x}}. \]

28. \[ x^{\log x} \frac{2\log x}{x} + (\log x)^x \left[ \frac{1}{\log x} + \log(\log x) \right]. \]

29. \[ \frac{dy}{dx} = x^{x^x} \cdot x^x \log x \left( 1 + \log x + \frac{1}{x \log x} \right). \]

30. \[ \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x} \]

31. \[ \frac{dy}{dx} = -\frac{1}{2} \]

33. \[ (\log x)^{\log x} \left[ \frac{1}{x} + \frac{\log(\log x)}{x} \right], \quad x > 1 \]

35. \[ \sec^2 y \tan y. \]
**POINTS TO REMEMBER**

- **Rate of Change:** Let \( y = f(x) \) be a function then the rate of change of \( y \) with respect to \( x \) is given by \( \frac{dy}{dx} = f'(x) \) where a quantity \( y \) varies with another quantity \( x \).

\[
\left. \frac{dy}{dx} \right|_{x=x_0} \quad \text{or} \quad f'(x_0)
\]

represents the rate of change of \( y \) w.r.t. \( x \) at \( x = x_0 \).

- If \( x = f(t) \) and \( y = g(t) \)

By chain rule

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{if} \quad \frac{dx}{dt} \neq 0.
\]

- (i) A function \( f(x) \) is said to be increasing (non-decreasing) on an interval \((a, b)\) if \( x_1 < x_2 \) in \((a, b)\) \( \Rightarrow f(x_1) \leq f(x_2) \quad \forall \quad x_1, x_2 \in (a, b) \). Alternatively if \( f'(x) \geq 0 \quad \forall \quad x \in (a, b) \), then \( f(x) \) is increasing function in \((a, b)\).

(ii) A function \( f(x) \) is said to be decreasing (non-increasing) on an interval \((a, b)\). If \( x_1 < x_2 \) in \((a, b) \) \( \Rightarrow f(x_1) \geq f(x_2) \quad \forall \quad x_1, x_2 \in (a, b) \). Alternatively if \( f'(x) \leq 0 \quad \forall \quad x \in (a, b) \), then \( f(x) \) is decreasing function in \((a, b)\).

- The equation of tangent at the point \((x_0, y_0)\) to a curve \( y = f(x) \) is given by

\[
y - y_0 = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} (x - x_0).
\]
where \( \left. \frac{dy}{dx} \right|_{(x_0, y_0)} \) = slope of the tangent at the point \((x_0, y_0)\).

(i) If \( \left. \frac{dy}{dx} \right|_{(x_0, y_0)} \) does not exist then tangent at \((x_0, y_0)\) is parallel to \(y\)-axis and its equation is \(x = x_0\).

(ii) If tangent at \(x = x_0\) is parallel to \(x\)-axis then \( \left. \frac{dy}{dx} \right|_{(x_0, y_0)} = 0 \) and its equation is \(y = y_0\).

- Slope of the normal to the curve at the point \((x_0, y_0)\) is given by \( \frac{-1}{\left. \frac{dy}{dx} \right|_{(x_0, y_0)}} \).
- Equation of the normal to the curve \(y = f(x)\) at a point \((x_0, y_0)\) is given by \(y - y_0 = -\frac{1}{\left. \frac{dy}{dx} \right|_{(x_0, y_0)}}(x - x_0)\).

- If \( \left. \frac{dy}{dx} \right|_{(x_0, y_0)} = 0 \), then equation of the normal is \(x = x_0\) and equation of the tangent is \(y = y_0\).
- If \( \left. \frac{dy}{dx} \right|_{(x_0, y_0)} \) does not exist, then the normal is parallel to \(x\)-axis and the equation of the normal is \(y = y_0\).

- Let \( y = f(x) \)
  \(\Delta x\) = the small increment in \(x\) and
  \(\Delta y\) be the increment in \(y\) corresponding to the increment in \(x\)

Then approximate change in \(y\) is given by

\[ \Delta y = \left( \frac{dy}{dx} \right) \Delta x \quad \text{or} \quad dy = f'(x) \Delta x \]

The approximate change in the value of \(f\) is given by

\[ f(x + \Delta x) = f(x) + f'(x) \Delta x \]
Let $f$ be a function. Let point $c$ be in the domain of the function $f$ at which either $f'(x) = 0$ or $f$ is not derivable is called a critical point of $f$.

First Derivative Test: Let $f$ be a function defined on an open interval $I$. Let $f$ be continuous at a critical point $c \in I$. Then if,

(i) $f'(x)$ changes sign from positive to negative as $x$ increases through $c$, then $c$ is called the point of the local maxima.

(ii) $f'(x)$ changes sign from negative to positive as $x$ increases through $c$, then $c$ is a point of local minima.

(iii) $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Such a point is called a point of inflexion.

Second Derivative Test: Let $f$ be a function defined on an interval $I$ and let $c \in I$. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. $f(c)$ is local maximum value of $f$.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. $f(c)$ is local minimum value of $f$.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

**VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)**

1. The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of perimeter of the square.

2. The radius of the circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

3. If the radius of a soap bubble is increasing at the rate of $\frac{1}{2}$ cm/sec. At what rate its volume is increasing when the radius is 1 cm.

4. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?
5. The total revenue in rupees received from the sale of \( x \) units of a product is given by
\[
R(x) = 13x^2 + 26x + 15. 
\]
Find the marginal revenue when \( x = 7 \).

6. Find the maximum and minimum values of the function \( f(x) = \sin 2x + 5 \).

7. Find the maximum and minimum values (if any) of the function
\[
f(x) = -|x - 1| + 7 \forall x \in \mathbb{R}.
\]

8. Find the value of ‘a’ for which the function \( f(x) = x^2 - 2ax + 6, x > 0 \) is strictly increasing.

9. Write the interval for which the function \( f(x) = \cos x, 0 \leq x \leq 2\pi \) is decreasing.

10. What is the interval for which the function \( f(x) = \frac{\log x}{x}, x \in (0, \infty) \) is increasing?

11. For which values of \( x \), the function \( y = x^4 - \frac{4}{3}x^3 \) is increasing?

12. Write the interval for which the function \( f(x) = \frac{1}{x} \) is strictly decreasing.

13. Find the sub-interval of the interval \((0, \pi/2)\) in which the function \( f(x) = \sin 3x \) is increasing.

14. Without using derivatives, find the maximum and minimum value of \( y = |3 \sin x + 1| \).

15. If \( f(x) = ax + \cos x \) is strictly increasing on \( \mathbb{R} \), find \( a \).

16. Write the interval in which the function \( f(x) = x^3 + 3x^2 + 64 \) is increasing.

17. What is the slope of the tangent to the curve \( f = x^3 - 5x + 3 \) at the point whose \( x \) co-ordinate is 2?

18. At what point on the curve \( y = x^2 \) does the tangent make an angle of \( 45^\circ \) with positive direction of the \( x \)-axis?

19. Find the point on the curve \( y = 3x^2 - 12x + 9 \) at which the tangent is parallel to \( x \)-axis.
20. What is the slope of the normal to the curve \( y = 5x^2 - 4 \sin x \) at \( x = 0 \).

21. Find the point on the curve \( y = 3x^2 + 4 \) at which the tangent is perpendicular to the line with slope \( \frac{1}{6} \).

22. Find the point on the curve \( y = x^2 \) where the slope of the tangent is equal to the \( y \) – co-ordinate.

23. If the curves \( y = 2e^x \) and \( y = ae^{-x} \) intersect orthogonally (cut at right angles), what is the value of \( a \)?

24. Find the slope of the normal to the curve \( y = 8x^2 - 3 \) at \( x = \frac{1}{4} \).

25. Find the rate of change of the total surface area of a cylinder of radius \( r \) and height \( h \) with respect to radius when height is equal to the radius of the base of cylinder.

26. Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing w.r.t. its radius when its radius is 3 cm?

27. For the curve \( y = (2x + 1)^3 \) find the rate of change of slope of the tangent at \( x = 1 \).

28. Find the slope of the normal to the curve

\[
x = 1 - a \sin \theta \quad ; \quad y = b \cos^2 \theta \quad \text{at} \quad \theta = \frac{\pi}{2}
\]

29. If a manufacturer’s total cost function is \( C(x) = 1000 + 40x + x^2 \), where \( x \) is the output, find the marginal cost for producing 20 units.

30. Find ‘\( a \)’ for which \( f(x) = a (x + \sin x) \) is strictly increasing on \( R \).

**SHORT ANSWER TYPE QUESTIONS (4 MARKS)**

31. A particle moves along the curve \( 6y = x^3 + 2 \). Find the points on the curve at which the \( y \) co-ordinate is changing 8 times as fast as the \( x \) co-ordinate.

32. A ladder 5 metres long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 cm/sec. How fast is its height on the wall decreasing when the foot of the ladder is 4 metres away from the wall?
33. A balloon which always remain spherical is being inflated by pumping in 900 cubic cm of a gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

34. A man 2 metres high walks at a uniform speed of 6 metres per minute away from a lamp post 5 metres high. Find the rate at which the length of his shadow increases.

35. Water is running out of a conical funnel at the rate of 5 cm³/sec. If the radius of the base of the funnel is 10 cm and altitude is 20 cm, find the rate at which the water level is dropping when it is 5 cm from the top.

36. The length \( x \) of a rectangle is decreasing at the rate of 2 cm/sec and the width \( y \) is increasing as the rate of 2 cm/sec when \( x = 12 \) cm and \( y = 5 \) cm. Find the rate of change of

(a) Perimeter
(b) Area of the rectangle.

37. Sand is pouring from a pipe at the rate of 12 c.c/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when height is 4 cm?

38. The area of an expanding rectangle is increasing at the rate of 48 cm²/sec. The length of the rectangle is always equal to the square of the breadth. At what rate is the length increasing at the instant when the breadth is 4.5 cm?

39. Find a point on the curve \( y = (x - 3)^2 \) where the tangent is parallel to the line joining the points (4, 1) and (3, 0).

40. Find the equation of all lines having slope zero which are tangents to the curve \( y = \frac{1}{x^2 - 2x + 3} \).

41. Prove that the curves \( x = y^2 \) and \( xy = k \) cut at right angles if \( 8k^2 = 1 \).

42. Find the equation of the normal at the point \((am^2, \ am^3)\) for the curve \( ay^2 = x^3 \).

43. Show that the curves \( 4x = y^2 \) and \( 4xy = k \) cut as right angles if \( k^2 = 512 \).

44. Find the equation of the tangent to the curve \( y = \sqrt{3x - 2} \) which is parallel to the line \( 4x - y + 5 = 0 \).
45. Find the equation of the tangent to the curve \( \sqrt{x} + \sqrt{y} = a \) at the point \( \left( \frac{a^2}{4}, \frac{a^2}{4} \right) \).

46. Find the points on the curve \( 4y = x^3 \) where slope of the tangent is \( \frac{16}{3} \).

47. Show that \( \frac{x}{a} + \frac{y}{b} = 1 \) touches the curve \( y = be^{-x/a} \) at the point where the curve crosses the y-axis.

48. Find the equation of the tangent to the curve given by \( x = 1 - \cos \theta \), \( y = \theta - \sin \theta \) at a point where \( \theta = \frac{\pi}{4} \).

49. Find the intervals in which the function \( f(x) = \log (1 + x) - \frac{x}{1 + x}, x > -1 \) is increasing or decreasing.

50. Find the intervals in which the function \( f(x) = x^3 - 12x^2 + 36x + 17 \) is
   (a) Increasing  (b) Decreasing.

51. Prove that the function \( f(x) = x^2 - x + 1 \) is neither increasing nor decreasing in \([0, 1]\).

52. Find the intervals on which the function \( f(x) = \frac{x}{x^2 + 1} \) is decreasing.

53. Prove that \( f(x) = \frac{x^3}{3} - x^2 + 9x, x \in [1, 2] \) is strictly increasing. Hence find the minimum value of \( f(x) \).

54. Find the intervals in which the function \( f(x) = \sin^4 x + \cos^4 x, 0 \leq x \leq \frac{\pi}{2} \) is increasing or decreasing.

55. Find the least value of 'a' such that the function \( f(x) = x^2 + ax + 1 \) is strictly increasing on \((1, 2)\).
56. Find the interval in which the function \( f(x) = 5x^2 - 3x^{-\frac{5}{2}} \), \( x > 0 \) is strictly decreasing.

57. Show that the function \( f(x) = \tan^{-1} (\sin x + \cos x) \), is strictly increasing on the interval \( \left( 0, \frac{\pi}{4} \right) \).

58. Show that the function \( f(x) = \cos \left( 2x + \frac{\pi}{4} \right) \) is strictly increasing on \( \left( \frac{3\pi}{8}, \frac{7\pi}{8} \right) \).

59. Show that the function \( f(x) = \frac{\sin x}{x} \) is strictly decreasing on \( \left( 0, \frac{\pi}{2} \right) \).

Using differentials, find the approximate value of (Q. No. 60 to 64).

60. \((0.009)^{\frac{1}{3}}\).

61. \((80)^{\frac{1}{4}}\).

62. \((0.0037)^{\frac{1}{2}}\).

63. \(\sqrt{0.037}\).

64. \(\sqrt{25.02}\).

65. Find the approximate value of \( f(5.001) \) where \( f(x) = x^3 - 7x^2 + 15 \).

66. Find the approximate value of \( f(3.02) \) where \( f(x) = 3x^2 + 5x + 3 \).

**LONG ANSWER TYPE QUESTIONS (6 MARKS)**

67. Show that of all rectangles inscribed in a given fixed circle, the square has the maximum area.

68. Find two positive numbers \( x \) and \( y \) such that their sum is 35 and the product \( x^2y^2 \) is maximum.

69. Show that of all the rectangles of given area, the square has the smallest perimeter.

70. Show that the right circular cone of least curved surface area and given volume has an altitude equal to \( \sqrt{2} \) times the radius of the base.
71. Show that the semi vertical angle of right circular cone of given surface area and maximum volume is \( \sin^{-1}\left(\frac{1}{3}\right) \).

72. A point on the hypotenuse of a triangle is at distance ‘a’ and ‘b’ from the sides of the triangle. Show that the minimum length of the hypotenuse is \( \left(\frac{2}{a^2} + \frac{2}{b^2}\right)^{\frac{3}{2}} \).

73. Prove that the volume of the largest cone that can be inscribed in a sphere of radius \( R \) is \( \frac{8}{27} \) of the volume of the sphere.

74. Find the interval in which the function \( f \) given by \( f(x) = \sin x + \cos x \), \( 0 \leq x \leq 2\pi \) is strictly increasing or strictly decreasing.

75. Find the intervals in which the function \( f(x) = (x + 1)^3 (x - 3)^3 \) is strictly increasing or strictly decreasing.

76. Find the local maximum and local minimum of \( f(x) = \sin 2x - x \), \( -\frac{\pi}{2} < x < \frac{\pi}{2} \).

77. Find the intervals in which the function \( f(x) = 2x^3 - 15x^2 + 36x + 1 \) is strictly increasing or decreasing. Also find the points on which the tangents are parallel to \( x \)-axis.

78. A solid is formed by a cylinder of radius \( r \) and height \( h \) together with two hemisphere of radius \( r \) attached at each end. It the volume of the solid is constant but radius \( r \) is increasing at the rate of \( \frac{1}{2\pi} \) metre/min. How fast must \( h \) (height) be changing when \( r \) and \( h \) are 10 metres.

79. Find the equation of the normal to the curve \( x = a (\cos \theta + \theta \sin \theta) ; y = a (\sin \theta - \theta \cos \theta) \) at the point \( \theta \) and show that its distance from the origin is \( a \).

80. For the curve \( y = 4x^3 - 2x^5 \), find all the points at which the tangent passes through the origin.

81. Find the equation of the normal to the curve \( x^2 = 4y \) which passes through the point \( (1, 2) \).
82. Find the equation of the tangents at the points where the curve \(2y = 3x^2 - 2x - 8\) cuts the \(x\)-axis and show that they make supplementary angles with the \(x\)-axis.

83. Find the equations of the tangent and normal to the hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) at the point \((x_0, y_0)\).

84. A window is in the form of a rectangle surmounted by an equilateral triangle. Given that the perimeter is 16 metres. Find the width of the window in order that the maximum amount of light may be admitted.

85. A jet of an enemy is flying along the curve \(y = x^2 + 2\). A soldier is placed at the point \((3, 2)\). What is the nearest distance between the soldier and the jet?

86. Find a point on the parabola \(y^2 = 4x\) which is nearest to the point \((2, -8)\).

87. A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum.

88. A window in the form of a rectangle is surmounted by a semi circular opening. The total perimeter of the window is 30 metres. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

89. An open box with square base is to be made out of a given iron sheet of area 27 sq. meter, show that the maximum value of the box is 13.5 cubic metres.

90. A wire of length 36 m is to be cut into two pieces. One of the two pieces is to be made into a square and other into a circle. What should be the length of two pieces so that the combined area of the square and the circle is minimum?

91. Show that the height of the cylinder of maximum volume which can be inscribed in a sphere of radius \(R\) is \(\frac{2R}{\sqrt{3}}\). Also find the maximum volume.

92. Show that the altitude of the right circular cone of maximum volume that can be inscribed is a sphere of radius \(r\) is \(\frac{4r}{3}\).
93. Prove that the surface area of solid cuboid of a square base and given volume is minimum, when it is a cube.

94. Show that the volume of the greatest cylinder which can be inscribed in a right circular cone of height \( h \) and semi-vertical angle \( \alpha \) is \( \frac{4}{27} \pi h^3 \tan^2 \alpha \).

95. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

96. A given quantity of metal is to be cast half cylinder with a rectangular box and semicircular ends. Show that the total surface area is minimum when the ratio of the length of cylinder to the diameter of its semicircular ends is \( \pi : (\pi + 2) \).

**ANSWERS**

1. 0.8 cm/sec.
2. 4.4 cm/sec.
3. \( 2\pi \) cm\(^3\)/sec.
4. \( 80\pi \) cm\(^2\)/sec.
5. Rs. 208.
7. Maximum value = 7, minimum value does not exist.
8. \( a \leq 0 \).
9. \([0, \pi]\)
10. \((0, e]\)
11. \(x \geq 1\)
12. \((-\infty, 0) \cup (0, \infty)\)
13. \(\left(0, \frac{\pi}{6}\right)\).
14. Maximum value = 4, minimum value = 0.
15. \(a > 1\).
16. \(R\)
17. 7
18. \(\left(\frac{1}{2}, \frac{1}{4}\right)\).
19. \((2, -3)\)
20. \(\frac{1}{4}\)
21. \((1, 7)\)
22. (0, 0), (2, 4)  23. \( \frac{1}{2} \)

24. \( \frac{1}{4} \)  25. \( 6\pi h \)

26. \( 2\pi r \text{ cm}^2/\text{cm}, \ 6\pi \text{ cm}^2/\text{cm} \)  27. 72

28. \( \frac{a}{2b} \)  29. Rs. 80.

30. \( a > 0 \).

31. (4, 11) and \( \left( -4, -\frac{31}{3} \right) \).  32. \( -\frac{8}{3} \text{ cm/sec.} \)

33. \( \frac{1}{\pi} \text{ cm/sec.} \)  34. 4 metres/minute

35. \( \frac{4}{45\pi} \text{ cm/sec.} \)  36. (a) 0 cm/sec., (b) 14 cm²/sec.

37. \( \frac{1}{48\pi} \text{ cm/sec.} \)  38. 7.11 cm/sec.

39. \( \begin{pmatrix} 7 & 1 \\ 2 & 4 \end{pmatrix} \)  40. \( y = \frac{1}{2} \).

42. \( 2x + 3my = am^2 (2 + 3m^2) \)

44. \( 48x - 24y = 23 \)

45. \( 2x + 2y = a^2 \)

46. \( \begin{pmatrix} 8 \\ 3 \end{pmatrix}, \begin{pmatrix} 128 \\ 27 \end{pmatrix}, \begin{pmatrix} -8 \\ 3 \end{pmatrix}, \begin{pmatrix} -128 \\ 27 \end{pmatrix} \).

48. \( (\sqrt{2} - 1)x - y = 2(\sqrt{2} - 1) - \frac{\pi}{4} \).

49. Increasing in \( (0, \infty) \), decreasing in \( (-1, 0) \).

50. Increasing in \( (-\infty, 2) \cup (6, \infty) \), Decreasing in \( (2, 6) \).
52. \((-\infty, -1)\) and \((1, \infty)\).

53. \(\frac{25}{3}\).

54. Increasing in \(\left(\frac{\pi}{4}, \frac{\pi}{2}\right)\) Decreasing in \(\left(0, \frac{\pi}{4}\right)\).

55. \(a = -2\).

56. Strictly decreasing in \((1, \infty)\).

60. 0.2083

61. 2.9907

62. 0.06083

63. 0.1923

64. 5.002

65. -34.995

66. 45.46

68. 25, 10

74. Strictly increasing in \(\left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right)\)

Strictly decreasing in \(\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)\).

75. Strictly increasing in \((1, 3) \cup (3, \infty)\)

Strictly decreasing in \((-\infty, -1) \cup (-1, 1)\).

76. Local maxima at \(x = \frac{\pi}{6}\)

Local max. value \(= \frac{\sqrt{3}}{2} - \frac{\pi}{6}\)

Local minima at \(x = -\frac{\pi}{6}\)

Local minimum value \(= -\frac{\sqrt{3}}{2} + \frac{\pi}{6}\)

77. Strictly increasing in \((-\infty, 2) \cup (3, \infty)\)

Strictly decreasing in \((2, 3)\).
Points are (2, 29) and (3, 28).

78. \(-\frac{3}{\pi}\) metres/min.

79. \(x + y \tan \theta - a \sec \theta = 0\).

80. (0, 0), (−1, −2) and (1, 2).

81. \(x + y = 3\)

82. \(5x - y - 10 = 0\) and \(15x + 3y + 20 = 0\)

83. \(\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1\), \(\frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0\).

84. \(\frac{16}{6-\sqrt{3}}\)

85. \(\sqrt{5}\)

86. (4, −4)

87. 4cm

88. \(\frac{60}{\pi+4}, \frac{30}{\pi+4}\)

89. \(\frac{144}{\pi+4}\) m, \(\frac{36\pi}{\pi+4}\) m.

90. \(\frac{4\pi R^3}{3\sqrt{3}}\)
CHAPTER 7
INTEGRALS

POINTS TO REMEMBER

- Integration is the reverse process of Differentiation.

- Let \( \frac{d}{dx} F(x) = f(x) \) then we write \( \int f(x) \, dx = F(x) + c \).

- These integrals are called indefinite integrals and \( c \) is called constant of integration.

- From geometrical point of view an indefinite integral is collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along \( y \)-axis.

STANDARD FORMULAE

1. \( \int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log |x| + c & n = -1 \end{cases} \)

2. \( \int (ax + b)^n \, dx = \begin{cases} \frac{(ax + b)^{n+1}}{(n+1)a} + c & n \neq -1 \\ \frac{1}{a} \log |ax + b| + c & n = -1 \end{cases} \)

3. \( \int \sin x \, dx = -\cos x + c \)  
4. \( \int \cos x \, dx = \sin x + c \)

5. \( \int \tan x \, dx = -\log |\cos x| + c = \log |\sec x| + c \)
6. \( \int \cot x \, dx = \log |\sin x| + c \)
7. \( \int \sec^2 x \, dx = \tan x + c \)
8. \( \int \csc^2 x \, dx = -\cot x + c \)
9. \( \int \sec x \cdot \tan x \, dx = \sec x + c \)
10. \( \int \csc x \cot x \, dx = -\csc x + c \)
11. \( \int \sec x \, dx = \log |\sec x + \tan x| + c \)
12. \( \int \csc x \, dx = \log |\csc x - \cot x| + c \)
13. \( \int e^x \, dx = e^x + c \)
14. \( \int a^x \, dx = \frac{a^x}{\log a} + c \)
15. \( \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + c, \, |x| < 1 \)
16. \( \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + c \)
17. \( \int \frac{1}{x \sqrt{x^2 - 1}} \, dx = \sec^{-1} x + c, \, |x| > 1 \)
18. \( \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + c \)
19. \( \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c \)
20. \( \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \)
21. \[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + c. \]

22. \[ \int \frac{1}{\sqrt{a^2 + x^2}} \, dx = \log \left| x + \sqrt{a^2 + x^2} \right| + c. \]

23. \[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c. \]

24. \[ \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c. \]

25. \[ \int \sqrt{a^2 + x^2} \, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log \left| x + \sqrt{a^2 + x^2} \right| + c. \]

26. \[ \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + c. \]

**RULES OF INTEGRATION**

1. \[ \int k f(x) \, dx = k \int f(x) \, dx. \]

2. \[ \int k \{ f(x) \pm g(x) \} \, dx = k \int f(x) \, dx \pm k \int g(x) \, dx. \]

2. \[ \int e^x \{ f(x) + f'(x) \} \, dx = e^x f(x) + c. \]

**INTEGRATION BY SUBSTITUTION**

1. \[ \int \frac{f'(x)}{f(x)} \, dx = \log |f(x)| + c. \]

2. \[ \int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c. \]
3. \[ \int \frac{f(x)}{[f(x)]^n} \, dx = \frac{(f(x))^{-n+1}}{-n + 1} + c. \]

**INTEGRATION BY PARTS**

\[ \int f(x) \cdot g(x) \, dx = f(x) \cdot \left[ \int g(x) \, dx \right] - \int f'(x) \cdot \left[ \int g(x) \, dx \right] \, dx. \]

**DEFINITE INTEGRALS**

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a), \text{ where } F(x) = \int f(x) \, dx. \]

**DEFINITE INTEGRAL AS A LIMIT OF SUMS.**

\[ \int_{a}^{b} f(x) \, dx = \lim_{h \to 0} \left( \sum_{r=1}^{n} f(a + rh) \right) \]

where \( h = \frac{b - a}{h} \). or \( \int_{a}^{b} f(x) \, dx = \lim_{h \to 0} \left[ h \sum_{r=1}^{n} f(a + rh) \right] \)

**PROPERTIES OF DEFINITE INTEGRAL**

1. \[ \int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx. \]
2. \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt. \]
3. \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx. \]
4. (i) \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(a + b - x) \, dx. \] (ii) \[ \int_{0}^{a} f(x) \, dx = \int_{0}^{a} f(a - x) \, dx. \]
5. \[ \int_{-a}^{a} f(x) \, dx = 0; \text{ if } f(x) \text{ is odd function.} \]

6. \[ \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx; \text{ if } f(x) \text{ is even function.} \]

7. \[ \int_{0}^{2a} f(x) \, dx = \begin{cases} 
\int_{0}^{a} f(x) \, dx, & \text{if } f(2a-x)=f(x) \\
0, & \text{if } f(2a-x)=-f(x) 
\end{cases} \]

**VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)**

Evaluate the following integrals

1. \[ \int_{-1}^{1} \left( \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} \right) \, dx. \]

2. \[ \int_{-1}^{1} e^{x^2} \, dx. \]

3. \[ \int_{-1}^{1} \frac{1}{1-\sin^2 x} \, dx. \]

4. \[ \int_{0}^{1} \left( e^{x^2} + x^8 + \frac{x^8}{8} \right) \, dx. \]

5. \[ \int_{-1}^{1} x^{99} \cos^4 x \, dx. \]

6. \[ \int_{0}^{1} \frac{1}{x \log x \log(\log x)} \, dx. \]

7. \[ \int_{0}^{\pi/2} \log \left( \frac{4 + 3 \sin x}{4 + 3 \cos x} \right) \, dx. \]

8. \[ \int_{0}^{\infty} \left( e^{a \log x} + e^{x \log a} \right) \, dx. \]

9. \[ \int_{-\pi/2}^{\pi/2} \left( \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} \right) \, dx. \]

10. \[ \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx. \]

11. \[ \int_{-\pi/2}^{\pi/2} \sqrt{10 - 4x + x^2} \, dx. \]

12. \[ \frac{d}{dx} \left[ \int_{-\pi/2}^{\pi/2} f(x) \, dx \right]. \]
13. \[\int \frac{1}{\sin^2 x \cos^2 x} \, dx\]

14. \[\int \frac{1}{\sqrt{x + \sqrt{x - 1}}} \, dx\]

15. \[\int e^{-\log x} \, dx\]

16. \[\int \frac{e^x}{a^x} \, dx\]

17. \[\int 2^x e^x \, dx\]

18. \[\int \frac{x}{\sqrt{x + 1}} \, dx\]

19. \[\int \frac{x}{(x + 1)^2} \, dx\]

20. \[\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx\]

21. \[\int \cos^2 \alpha \, dx\]

22. \[\int \frac{1}{x \cos \alpha + 1} \, dx\]

23. \[\int \sec x \log(\sec x + \tan x) \, dx\]

24. \[\int \frac{1}{\cos \alpha + x \sin \alpha} \, dx\]

25. \[\int \cot x \log |\sin x| \, dx\]

26. \[\int \left(\frac{x - 1}{x}\right)^3 \, dx\]

27. \[\int \frac{1}{x(2 + 3 \log x)} \, dx\]

28. \[\int \frac{1 - \sin x}{x + \cos x} \, dx\]

29. \[\int \frac{1 - \cos x}{\sin x} \, dx\]

30. \[\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} \, dx\]

31. \[\int \frac{(x + 1)(x + \log x)}{x} \, dx\]

32. \[\int \left(\sqrt{ax} - \frac{1}{\sqrt{ax}}\right)^2 \, dx\]

33. \[\int_0^\pi \cos x \, dx\]

34. \[\int_0^2 [x] \, dx \text{ where } [x] \text{ is greatest integer function.}\]
35. \[ \int \frac{1}{\sqrt{9 - 4x^2}} \, dx \]

36. \[ \int_{a}^{b} \frac{f(x)}{f(x) + f(a + b - x)} \, dx \]

37. \[ \int_{-2}^{2} |x| \, dx \]

38. \[ \int_{-1}^{1} x |x| \, dx \]

39. If \( \int_{0}^{a} \frac{1}{1 + x^2} = \frac{\pi}{4} \), then what is value of \( a \).

40. \[ \int_{a}^{b} f(x) \, dx + \int_{a}^{b} f(x) \, dx \]

41. \[ \int e^{\log(x + 1) - \log x} \, dx \]

42. \[ \int \frac{\sin x}{\sin 2x} \, dx \]

43. \[ \int \sin x \sin 2x \, dx \]

44. \[ \int_{\pi/4}^{\pi/4} |\sin x| \, dx \]

45. \[ \int_{a}^{b} f(x) \, dx + \int_{a}^{b} f(a + b - x) \, dx \]

46. \[ \int \frac{1}{\sec x + \tan x} \, dx \]

47. \[ \int \frac{\sin^2 x}{1 + \cos x} \, dx \]

48. \[ \int \frac{1 - \tan x}{1 + \tan x} \, dx \]

49. \[ \int \frac{a^x + b^x}{c^x} \, dx \]

**SHORT ANSWER TYPE QUESTIONS (4 MARKS)**

50. (i) \[ \int \frac{x \cosec \left( \tan^{-1} \frac{x}{2} \right)}{1 + x^4} \, dx \]

(ii) \[ \int \frac{\sqrt{x + 1} - \sqrt{x - 1}}{\sqrt{x + 1} + \sqrt{x - 1}} \, dx \]

(iii) \[ \int \frac{1}{\sin (x - a) \sin (x - b)} \, dx \]

(iv) \[ \int \frac{\cos (x + a)}{\cos (x - a)} \, dx \]
(v) \[ \int \cos x \cos 2x \cos 3x \, dx \]

(vi) \[ \int \cos^5 x \, dx \]

(vii) \[ \int \sin^2 x \cos^4 x \, dx \]

(viii) \[ \int \cot^3 x \cosec^4 x \, dx \]

(ix) \[ \int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} \, dx \]

[Hint : put \( a^2 \sin^2 x + b^2 \cos^2 x = t \) or \( t^2 \)]

(x) \[ \int \frac{1}{\cos^3 x \cos (x + a)} \, dx \]

[Hint : Take \( \sec^2 x \) as numerator]

(xi) \[ \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} \, dx \]

(xii) \[ \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} \, dx \]

51. Evaluate :

(i) \[ \int \frac{x}{x^4 + x^2 + 1} \, dx \]

[Hint : put \( x^2 = t \)]

(ii) \[ \int \frac{1}{x \left[ 6 (\log x)^2 + 7 \log x + 2 \right]} \, dx \]

[Hint : put \( \log x = t \)]

(iii) \[ \int \frac{dx}{1 + x - x^2} \]

(iv) \[ \int \frac{1}{\sqrt{9 + 8x - x^2}} \, dx \]

(v) \[ \int \frac{1}{\sqrt{(x - a)(x - b)}} \, dx \]

(vi) \[ \int \frac{5x - 2}{3x^2 + 2x + 1} \, dx \]

(vii) \[ \int \frac{x^2}{x^2 + 6x + 12} \, dx \]

(viii) \[ \int \frac{x + 2}{\sqrt{4x - x^2}} \, dx \]

(ix) \[ \int \frac{x}{x \sqrt{1 + x - x^2}} \, dx \]
(x) \[ \int (3x - 2) \sqrt{x^2 + x + 1} \, dx \]

(xi) \[ \int \sec x - 1 \, dx. \]

[Hint : Multiply and divide by \( \sqrt{\sec x + 1} \)]

52. Evaluate :

(i) \[ \int \frac{dx}{x (x^2 + 1)} \]

(ii) \[ \int \frac{\sin x}{(1 + \cos x)(2 + 3 \cos x)} \, dx. \]

(iii) \[ \int \frac{\sin \theta \cos \theta}{\cos^2 \theta - \cos \theta - 2} \, d\theta. \]

(iv) \[ \int \frac{x - 1}{(x + 1)(x - 2)(x + 3)} \, dx. \]

(v) \[ \int \frac{x^2 + x + 2}{(x - 2)(x - 1)} \, dx. \]

(vi) \[ \int \frac{(x^2 + 1)(x^2 + 2)}{(x^3 + 3)(x^2 + 4)} \, dx. \] [Hint : \( x^2 = t \)]

(vii) \[ \int \frac{dx}{(2x + 1)(x^2 + 4)}. \]

(viii) \[ \int \frac{x^2 - 1}{x^4 + x^2 + 1} \, dx. \]

(ix) \[ \int \sqrt{\tan x} \, dx. \]
53. Evaluate:

(i) \[ \int x^5 \sin x^3 \, dx. \]

(ii) \[ \int \sec^3 x \, dx. \]

[Hint: Write \( \sec^3 x = \sec x \cdot \sec^2 x \) and take \( \sec x \) as the first function]

(iii) \[ \int e^{ax} \cos (bx + c) \, dx. \]

(iv) \[ \int \sin^{-1} \left( \frac{6x}{1 + 9x^2} \right) \, dx. \]

[Hint: put \( 3x = \tan \theta \)]

(v) \[ \int \cos \sqrt{x} \, dx. \]

(vi) \[ \int x^3 \tan^{-1} x \, dx. \]

(vii) \[ \int e^{2x} \left( \frac{1 + \sin 2x}{1 + \cos 2x} \right) \, dx. \]

(viii) \[ \int e^x \left( \frac{x - 1}{2x^2} \right) \, dx. \]

(ix) \[ \int \sqrt{2ax - x^2} \, dx. \]

(x) \[ \int e^x \left( \frac{x^2 + 1}{(x + 1)^2} \right) \, dx. \]

(xi) \[ \int e^x \left( \frac{2 + \sin 2x}{1 + \cos 2x} \right) \, dx. \]

(xii) \[ \int \left( \log(\log x) + \frac{1}{(\log x)^2} \right) \, dx. \]

[Hint: put \( \log x = t \Rightarrow x = e^t \)]

(xiii) \[ \int (6x + 5) \sqrt{6 + x - x^2} \, dx. \]

(xiv) \[ \int \frac{1}{x^3 + 1} \, dx. \]

(xv) \[ \int (2x - 5) \sqrt{x^2 - 4x + 3} \, dx. \]

(xvi) \[ \int \sqrt{x^2 - 4x + 8} \, dx. \]
54. Evaluate the following definite integrals:

(i) \[ \int_{0}^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} \, dx. \]

(ii) \[ \int_{0}^{\frac{\pi}{2}} \cos 2x \log \sin x \, dx. \]

(iii) \[ \int_{0}^{1} \frac{x}{\sqrt{1 - x^2}} \, dx. \]

(iv) \[ \int_{0}^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1 - x^2)^{3/2}} \, dx. \]

[Hint: put \( x^2 = t \)]

(v) \[ \int_{0}^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \, dx. \]

(vi) \[ \int_{0}^{1} \frac{5x^2}{x^2 + 4x + 3} \, dx. \]

(vii) \[ \int_{0}^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x} \, dx. \]

[Hint: Write \( \frac{x + \sin x}{1 + \cos x} \) as \( \frac{x}{1 + \cos x} + \frac{\sin x}{1 + \cos x} \)]

55. Evaluate:

(i) \[ \int_{1}^{3} |x - 1| + |x - 2| + |x - 3| \, dx. \]

(ii) \[ \int_{0}^{\pi} \frac{x}{1 + \sin x} \, dx. \]

(iii) \[ \int_{-1}^{1} e^{\tan^{-1} x} \left[ \frac{1 + x + x^2}{1 + x^2} \right] \, dx. \]

(iv) \[ \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx. \]

(v) \[ \int_{-2}^{2} f(x) \, dx \]

where \( f(x) = \begin{cases} 2x - x^3 & \text{when } -2 \leq x < -1 \\ x^3 - 3x + 2 & \text{when } -1 \leq x < 1 \\ 3x - 2 & \text{when } 1 \leq x < 2. \end{cases} \)

[Hint: \( \int_{-2}^{2} f(x) \, dx = \int_{-2}^{-1} f(x) \, dx + \int_{-1}^{1} f(x) \, dx + \int_{1}^{2} f(x) \, dx \)]
(vi) \[ \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} \, dx. \]

(vii) \[ \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx. \]

[Hint: Use \( f(x) \, dx = f(a-x) \, dx \)]

56. Evaluate the following integrals

(i) \[ \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} \]

(ii) \[ \int_0^1 \sin^{-1} \left( \frac{2x}{1 + x^2} \right) \, dx. \]

(iii) \[ \int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} \, dx. \]

(iv) \[ \int_0^\pi \frac{x \tan x}{\sec x \cos x} \, dx. \]

(v) \[ \int_{-a}^a \frac{\sqrt{a-x}}{\sqrt{a+x}} \, dx. \]

57. \[ \int_1^6 [ |x-2| + |x-4| - |x-5| ] \, dx \]

58. \[ \int e^{\log x + \log \sin x} \, dx. \]

59. Evaluate

(i) \[ \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} \, dx, \quad x \in [0, 1] \]
60. Evaluate the following integrals:

(i) \( \int \frac{x^5 + 4}{x^5 - x} \, dx \)

(ii) \( \int \frac{dx}{(x - 1)(x^2 + 4)} \, dx \)

(iii) \( \int \frac{2x^3}{(x + 1)(x - 3)^2} \, dx \)

(iv) \( \int \frac{x^4}{x^4 - 16} \, dx \)
(v) \[ \int_{0}^{\frac{\pi}{2}} \left( \sqrt{\tan x} + \sqrt{\cot x} \right) \, dx \]

(vi) \[ \int \frac{1}{x^4 + 1} \, dx \]

(vii) \[ \int_{0}^{x} \frac{x \tan^{-1} x}{\left(1 + x^2\right)^2} \, dx \]

61. Evaluate the following integrals as limit of sums:

(i) \[ \int_{2}^{4} (2x + 1) \, dx \]

(ii) \[ \int_{0}^{2} \left( x^2 + 3 \right) \, dx \]

(iii) \[ \int_{1}^{3} \left( 3x^2 - 2x + 4 \right) \, dx \]

(iv) \[ \int_{0}^{4} \left( 3x^2 + e^{2x} \right) \, dx \]

(v) \[ \int_{2}^{5} (x^2 + 3x) \, dx \]

62. Evaluate

(i) \[ \int_{0}^{1} \cot^{-1} \left( 1 - x + x^2 \right) \, dx \]

(ii) \[ \int \frac{dx}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} \]

(iii) \[ \int_{0}^{1} \frac{\log (1 + x)}{1 + x^2} \, dx \]

(iv) \[ \int_{0}^{\frac{\pi}{2}} \left( 2 \log \sin x - \log \sin 2x \right) \, dx \]

63. \[ \int \frac{1}{\sin x + \sin 2x} \, dx \]

64. \[ \int_{0}^{\frac{\pi}{2}} \frac{(3 \sin \theta - 2 \cos \theta)}{5 - \cos^2 \theta - 4 \sin \theta} \, d\theta \]
65. \[ \int_0^1 x (\tan^{-1} x)^2 \, dx \]

66. \[ \int e^{2x} \cos 3x \, dx \]

67. \[ \int_{0}^{\pi/2} \log \sin x \, dx \]

**ANSWERS**

1. \[ \frac{\pi}{2} x + c. \]

2. \[ 2e - 2 \]

3. \[ \tan x + c. \]

4. \[ \frac{8^x}{\log 8} + \frac{x^9}{9} + 8\log |x| + \frac{x^2}{16} + c. \]

5. \[ 0 \]

6. \[ \log |\log |\log x|| + c \]

7. \[ 0 \]

8. \[ \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c \]

9. \[ \tan x + c \]

10. \[ 0 \]

11. \[ \frac{(x - 2)\sqrt{x^2 - 4x + 10}}{2} + 3\log |x - 2 + \sqrt{x^2 - 4x + 10}| + c \]

12. \[ f(x) + c \]

13. \[ \tan x - \cot x + c \]

14. \[ \frac{2}{3}x^{3/2} - \frac{2}{3}(x - 1)^{3/2} + c \]

15. \[ \log |x| + c \]

16. \[ \left(\frac{e}{a}\right)^x / \log(e/a) + c \]

17. \[ \frac{2^x e^x}{\log(2e)} + c \]

18. \[ \frac{2}{3}(x + 1)^{3/2} - 2(x + 1)^{1/2} + c. \]

19. \[ \log|x + 1| + \frac{1}{x + 1} + c. \]

20. \[ 2e^{\sqrt{x}} + c \]

21. \[ x \cos^2 \alpha + c \]

22. \[ \frac{\log |x \cos \alpha + 1|}{\cos \alpha} + c. \]
23. \[
\frac{(\log|\sec x + \tan x|)^2}{2} + c
\]
24. \[
\frac{\log|\cos \alpha + x \sin \alpha|}{\sin \alpha} + c
\]
25. \[
\frac{(\log|\sin x|)^2}{2} + c
\]
26. \[
\frac{x^4}{4} + \frac{1}{2}x^2 - \frac{3x^2}{2} + 3\log |x| + c.
\]
27. \[
\frac{1}{3}\log|2 + 3\log x| + c.
\]
28. \[
\log |x + \cos x| + c.
\]
29. \[
2\log |\sec \frac{x}{2}| + c.
\]
30. \[
\frac{1}{e}\log|x^e + e^x| + c.
\]
31. \[
\frac{(x + \log x)^2}{2} + c
\]
32. \[
a\frac{x^2}{2} + \frac{\log|x|}{a} - 2x + c.
\]
33. 0
34. 1
35. \[
\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c
\]
36. \[
\frac{b-a}{2}
\]
37. -1
38. 0
39. 1
40. 0
41. \[
x + \log x + c.
\]
42. \[
\frac{1}{2}\log|\sec x + \tan x| + c.
\]
43. \[
-\frac{1}{2}\left(\frac{\sin 3x}{3} - \sin x\right) + c \text{ or } \frac{2}{3}\sin^3 x + c
\]
44. \[
2 - \sqrt{5}
\]
45. 0
46. \[
\log |1 + \sin x| + c
\]
47. \[
\log |\cos x + \sin x| + c
\]
48. \[
\log |\cos x + \sin x| + c
\]
49. \[
\frac{(a/c)^x}{\log|a/c|} + \frac{(b/c)^x}{\log|b/c|} + c,
\]
50. (i) \[ \frac{1}{2} \log \left[ \csc \left( \tan^{-1} x^2 \right) - \frac{1}{x^2} \right] + c. \]

(ii) \[ \frac{1}{2} \left( x^2 - x \sqrt{x^2 - 1} \right) + \frac{1}{2} \log \left| x + \sqrt{x^2 - 1} \right| + c. \]

(iii) \[ \frac{1}{\sin (a - b)} \log \left| \frac{\sin(x - a)}{\sin(x - b)} \right| + c \]

(iv) \[ x \cos 2a - \sin 2a \log |\sec(x - a)| + c. \]

(v) \[ \frac{1}{48} \left[ 12x + 6 \sin 2x + 3 \sin 4x + 2 \sin 6x \right] + c. \]

(vi) \[ \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c. \]

(vii) \[ \frac{1}{32} \left[ 2x + \frac{1}{2} \sin 2x - \frac{1}{2} \sin 4x - \frac{1}{6} \sin 6x \right] + c. \]

(viii) \[ -\left( \cot^6 x + \cot^4 x \right) = c. \]

(ix) \[ \frac{1}{a^2 - b^2} \sqrt{a^2 \sin^2 x + b^2 \cos^2 x} + c \]

(x) \[ -2 \csc a \cos a - \tan x \sin a + c. \]

(xi) \[ \tan x - \cot x - 3x + c. \]

(xii) \[ \sin^{-1} (\sin x - \cos x) + c. \]

51. (i) \[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x^2 + 1}{\sqrt{3}} \right) + c. \]

(ii) \[ \log \left| \frac{2 \log x + 1}{3 \log x + 2} \right| + C \]
(iii) \[ \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x} \right| + c \]

(iv) \[ \sin^{-1} \left( \frac{x - 4}{5} \right) + c. \]

(v) \[ 2 \log \left| \sqrt{x - a} + \sqrt{x - b} \right| + c \]

(vi) \[ \frac{5}{6} \log \left| 3x^2 + 2x + 1 \right| + \frac{(-11)}{3\sqrt{2}} \tan^{-1} \left( \frac{3x + 1}{\sqrt{2}} \right) + c \]

(vii) \[ x - 3 \log \left| x^2 + 6x + 12 \right| + 2\sqrt{3} \tan^{-1} \left( \frac{x + 3}{\sqrt{3}} \right) + c \]

(viii) \[ -\sqrt{4x - x^2} + 4 \sin^{-1} \left( \frac{x - 2}{2} \right) + c \]

(ix) \[ \frac{-1}{3} \left( 1 + x - x^2 \right)^{\frac{3}{2}} + \frac{1}{8} (2x - 1) \sqrt{1 + x - x^2} + \frac{5}{16} \sin^{-1} \left( \frac{2x - 1}{\sqrt{5}} \right) + c \]

(x) \[ \left( x^2 + x + 1 \right)^{\frac{3}{2}} - \frac{7}{2} \left[ \left( \frac{2x + 1}{4} \right) \sqrt{x^2 + x + 1} + \frac{3}{8} \log \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| \right] + c \]

(xi) \[-\log \left| \cos x + \frac{1}{2} + \sqrt{\cos^2 x + \cos x} \right| + c \]

52. (i) \[ \frac{1}{7} \log \left| \frac{x^7}{x^7 + 1} \right| + c \]
(ii) \[ \log \left| \frac{1 + \cos x}{2 + 3 \cos x} \right| + c \]

(iii) \( \frac{-2}{3} \log |\cos \theta - 2| - \frac{1}{3} \log |1 + \cos \theta| + c. \)

(iv) \( \frac{1}{3} \log |x + 1| + \frac{1}{15} \log |x - 2| - \frac{2}{5} \log |x + 3| + c \)

(v) \( x + 4 \log \left( \frac{(x - 2)^2}{x - 1} \right) + c \)

(vi) \( x + \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) - 3 \tan^{-1} \left( \frac{x}{2} \right) + c \)

(vii) \( \frac{2}{17} \log |2x + 1| - \frac{1}{17} \log |x^2 + 4| + \frac{1}{34} \tan^{-1} \frac{x}{2} + c \)

(viii) \( \frac{1}{2} \log \left| \frac{\alpha - x + 1}{\alpha^2 + x + 1} \right| + c \)

(ix) \( \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right| + c \)

53. (i) \( \frac{1}{3} \left[ -x^3 \cos x^3 + \sin x^3 \right] + c \)

(ii) \( \frac{1}{2} \left[ \sec x \tan x + \log |\sec x + \tan x| \right] + c \)

(iii) \( e^{ax} \left[ a \cos (bx + c) + b \sin (bx + c) \right] + c_1 \)

(iv) \( 2x \tan^{-1} 3x - \frac{1}{3} \log |1 + 9x^2| + c \)
(v) \[2 \left(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}\right) + c\]

(vi) \[\left(\frac{x^4 - 1}{4}\right) \tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} + c.\]

(vii) \[\frac{1}{2} e^{2x} \tan x + c.\]

(viii) \[e^x + c.\]

(ix) \[\frac{x - a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x - a}{a}\right) + c\]

(x) \[e^x \left(\frac{x - 1}{x + 1}\right) + c.\]

(xi) \[e^x \tan x + c.\]

(xii) \[x \log |\log x| - \frac{x}{\log x} + c.\]

(xiii) \[-2 \left(6 + x - x^2 \right)^{3/2} + 8 \left[\frac{2x - 1}{4} \sqrt{6 + x - x^2} + \frac{25}{8} \sin^{-1} \left(\frac{2x - 1}{5}\right)\right] + c\]

(xiv) \[\frac{1}{3} \log |x + 1| - \frac{1}{6} \log |x^2 - x + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x - 1}{\sqrt{3}}\right) + c\]

(xv) \[\frac{2}{3} \left(x^2 - 4x + 3\right)^{3/2} - \left(\frac{x - 2}{2}\right) \sqrt{x^2 - 4x + 3} + \frac{1}{2} \log |x - 2 + \sqrt{x^2 - 4x + 3}| + c\]

(xvi) \[\left(\frac{x - 2}{2}\right) \sqrt{x^2 - 4x + 8} + 2 \log |(x - 2) + \sqrt{x^2 - 4x + 8}| + c\]
54. (i) $\frac{1}{20} \log 3$. (ii) $-\frac{\pi}{4}$

(iii) $\frac{\pi}{4} - \frac{1}{2}$. (iv) $\frac{\pi}{4} - \frac{1}{2} \log 2$.

(v) $\frac{\pi}{2}$.

(vi) $5 - 10 \log \frac{15}{8} + \frac{25}{2} \log \left(\frac{6}{5}\right)$.

(vii) $\frac{\pi}{2}$.

55. (i) 5. (ii) $\pi$.

(iii) $e^\frac{\pi}{4} + e^{-\frac{\pi}{4}}$. (iv) $\frac{1}{4} \pi^2$.

(v) $\frac{29}{4}$. (vi) $\frac{\pi^2}{16}$.

(vii) $\frac{\pi^2}{2ab}$.

56. (i) $\frac{\pi}{12}$. (ii) $\frac{\pi}{2} - \log 2$.

(iii) $\frac{\pi}{2}$. (iv) $\frac{\pi^2}{4}$

(v) $a\pi$.

57. $\frac{13}{2}$

58. $-x \cos x + \sin x + c$. 
59. 
(i) \( \frac{2(2x - 1)}{\pi} \sin^{-1} \sqrt{x} + \frac{2\sqrt{x - x^2}}{\pi} - x + c \)

(ii) \( -2\sqrt{1 - x} + \cos^{-1} \sqrt{x} + \sqrt{x - x^2} + c \)

(iii) \( -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{3/2} \left[ \log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3}\right] + c \)

(iv) \( \frac{\sin x - x \cos x}{x \sin x + \cos x} + c \)

(v) \( (x + a) \tan^{-1} \sqrt[3]{\frac{x}{a}} - \sqrt{a}x + c \) 

(vi) \( 2\sin^{-1} \frac{\sqrt{3} - 1}{2} \)

(vii) 0 

(viii) \( \frac{3}{\pi} + \frac{1}{\pi^2} \)

(ix) \( \cos 2a (x + a) - (\sin 2a) \log |\sin (x + a)| + c \)

(x) \( I = -\frac{4}{5} \log |x^2 + 4| + \frac{9}{5} \log |x^2 + 9| + c \)

(xi) 1

60. 
(i) \( x - 4 \log |x| + \frac{5}{4} \log |x - 1| + \frac{3}{4} \log |x + 1| + \log |x^2 + 1| - \frac{1}{2} \tan^{-1} x + c. \)

(ii) \( \frac{1}{5} \log |x - 1| - \frac{1}{10} \log |x^2 + 4| = \frac{1}{10} \tan^{-1} \left(\frac{x}{2}\right) + c. \)

(iii) \( 2x - \frac{1}{8} \log |x + 1| + \frac{81}{8} \log |x - 3| - \frac{27}{2(x - 3)} + c. \)

(iv) \( x + \frac{1}{2} \log \left|\frac{x - 2}{x + 2}\right| - \tan^{-1} \left(\frac{x}{2}\right) + c. \)
(v) \( \pi/\sqrt{2} \).

(vi) \( \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log \left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c \)

(vii) \( \pi/8 \).

61. (i) 14. (ii) \( \frac{26}{3} \).

(iii) 26. (iv) \( \frac{1}{2} \left( 127 + e^8 \right) \).

(v) \( \frac{141}{2} \).

62. (i) \( \frac{\pi}{2} - \log 2 \) (ii) \( \frac{1}{5} \log \left|\frac{\tan x - 2}{2\tan x + 1}\right| + c \)

(iii) \( \frac{\pi}{8} \log 2. \) (iv) \( \frac{\pi}{2} \log \left(\frac{1}{2}\right) \).

63. \( \frac{1}{6} \log|1 - \cos x| + \frac{1}{2} \log|1 + \cos x| - \frac{2}{3} \log|1 + 2 \cos x| + c \).

64. \( 3 \log|2 - \sin \theta| + \frac{4}{2 - \sin \theta} + c. \)

65. \( \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2 \)

66. \( \frac{e^{2x}}{13}(2 \cos 3x + 3 \sin 3x) + c \).

67. \( -\frac{\pi}{2} \log 2 \)
CHAPTER 8
APPLICATIONS OF INTEGRALS

POINTS TO REMEMBER

AREA OF BOUNDED REGION

- Area bounded by the curve \( y = f(x) \), the \( x \) axis and between the ordinates, \( x = a \) and \( x = b \) is given by

\[
\text{Area} = \left| \int_{a}^{b} f(x) \, dx \right|
\]

- Area bounded by the curve \( x = f(y) \) the \( y \)-axis and between abscissas, \( y = c \) and \( y = d \) is given by

\[
\text{Area} = \left| \int_{c}^{d} f(y) \, dy \right|
\]
• Area bounded by two curves \( y = f(x) \) and \( y = g(x) \) such that \( 0 \leq g(x) \leq f(x) \) for all \( x \in [a, b] \) and between the ordinates \( x = a \) and \( x = b \) is given by

\[
\text{Area} = \int_a^b [f(x) - g(x)] \, dx
\]

• Required Area

\[
\text{Required Area} = \int_a^k f(x) \, dx + \int_k^b f(x) \, dx.
\]
4. Find the area of region in the first quadrant enclosed by $x$–axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

5. Find the area of region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$

6. Prove that the curve $y = x^2$ and, $x = y^2$ divide the square bounded by $x = 0, y = 0, x = 1, y = 1$ into three equal parts.

7. Find smaller of the two areas enclosed between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $bx + ay = ab$.

8. Find the common area bounded by the circles $x^2 + y^2 = 4$ and $(x – 2)^2 + y^2 = 4$.

9. Using integration, find the area of the region bounded by the triangle whose vertices are
   (a) $(-1, 0), (1, 3)$ and $(3, 2)$
   (b) $(-2, 2), (0, 5)$ and $(3, 2)$

10. Using integration, find the area bounded by the lines.
    (i) $x + 2y = 2, \ y - x = 1$ and $2x + y - 7 = 0$
    (ii) $y = 4x + 5, \ y = 5 - x$ and $4y - x = 5$.

11. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

12. Find the area of the region bounded by
    $y = |x - 1|$ and $y = 1$.

13. Find the area enclosed by the curve $y = \sin x$ between $x = 0$ and $x = \frac{3\pi}{2}$ and $x$-axis.

14. Find the area bounded by semi circle $y = \sqrt{25 - x^2}$ and $x$-axis.

15. Find area of region given by $\{(x, y) : x^2 \leq y \leq |x|\}$.

16. Find area of smaller region bounded by ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and straight line $2x + 3y = 6$. 
17. Find the area of region bounded by the curve \( x^2 = 4y \) and line \( x = 4y - 2 \).

18. Using integration find the area of region in first quadrant enclosed by \( x \)-axis, the line \( x = \sqrt{3}y \) and the circle \( x^2 + y^2 = 4 \).

19. Find smaller of two areas bounded by \( y = |x| \) and \( x^2 + y^2 = 8 \).

20. Find the area lying above \( x \)-axis and included between the circle \( x^2 + y^2 = 8x \) and inside the parabola \( y^2 = 4x \).

21. Using integration, find the area enclosed by the curve \( y = \cos x \), \( y = \sin x \) and \( x \)-axis in the interval \( \left( 0, \frac{\pi}{2} \right) \).

22. Sketch the graph \( y = |x - 5| \). Evaluate \( \int_{0}^{6} |x - 5| \, dx \).

23. Find the area enclosed between \( y = 4x \) and \( x^2 = 6y \).

24. Using integration, find the area of the following region:

\[ \{(x, y) : |x - 1| \leq y \leq \sqrt{5 - x^2} \} \]

25. Using integration, find the area of the triangle formed by positive \( x \)-axis and tangent and normal to the circle \( x^2 + y^2 = 4 \) at \((1, \sqrt{3})\).

26. Using integration, find the area of the region bounded by the line \( x - y + 2 = 0 \), the curve \( x = \sqrt{y} \) and \( y \)-axis.

**Answers**

1. \( \pi a^2 \) sq. units.

2. \( \left( \frac{25}{4} \pi - \frac{1}{2} \right) \) sq. units.

3. \( \pi ab \) sq. units

4. \( 4\pi \) sq. units

5. \( \frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \left( \frac{1}{3} \right) \) sq. units

6. \( \frac{(\pi - 2)ab}{4} \) sq. units

7. \( \frac{(\pi - 2)ab}{4} \) sq. units
8. \( \left( \frac{8\pi}{3} - 2\sqrt{3} \right) \) sq. units
9. (a) 4 sq. units (b) \( \frac{15}{2} \) sq. units
10. (a) 6 sq. units [Hint. Coordinate of vertices are (0, 1) (2, 3) (4, – 1)]
    (b) \( \frac{15}{2} \) sq. units
    [Hint : Coordinate of vertices are (– 1, 1) (0, 5) (3, 2)]
11. \( \left( \frac{\pi}{4} - \frac{1}{2} \right) \) sq. units
12. 1 sq. units
13. 3 sq. units
14. \( \frac{25}{2} \pi \) sq. units
15. \( \frac{1}{3} \) sq. units
16. \( \frac{3}{2} (\pi - 2) \) sq. units
17. \( \frac{9}{8} \) sq. units
18. \( \frac{\pi}{3} \) sq. units
19. \( 2\pi \) sq. units.
20. \( \frac{4}{3} (8 + 3\pi) \) sq. units
21. \( (2 - \sqrt{2}) \) sq. units.
22. 13 sq. units.
23. 384 sq. units.
24. \( \left( \frac{5\pi}{4} - \frac{1}{2} \right) \) sq. units
25. \( 2\sqrt{3} \) sq. units
26. \( \frac{10}{3} \) sq. units
CHAPTER 9

DIFFERENTIAL EQUATIONS

POINTS TO REMEMBER

- **Differential Equation**: Equation containing derivatives of a dependant variable with respect to an independent variable is called differential equation.

- **Order of a Differential Equation**: The order of a differential equation is defined to be the order of the highest order derivative occurring in the differential equation.

- **Degree of a Differential Equation**: Highest power of highest order derivative involved in the equation is called degree of differential equation where equation is a polynomial equation in differential coefficients.

- **Formation of a Differential Equation**: We differentiate the family of curves as many times as the number of arbitrary constant in the given family of curves. Now eliminate the arbitrary constants from these equations. After elimination the equation obtained is differential equation.

- **Solution of Differential Equation**
  
  (i) **Variable Separable Method**

  \[
  \frac{dy}{dx} = f(x, y)
  \]

  We separate the variables and get

  \[f(x)dx = g(y)dy\]

  Then \[\int f(x)dx = \int g(y)dy + c\] is the required solutions.

(ii) **Homogenous Differential Equation**: A differential equation of the form

  \[\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}\]

  where \(f(x, y)\) and \(g(x, y)\) are both
homogeneous functions of the same degree in \(x\) and \(y\) i.e., of the form \(\frac{dy}{dx} = F\left(\frac{y}{x}\right)\) is called a homogeneous differential equation.

For solving this type of equations we substitute \(y = vx\) and then \(\frac{dy}{dx} = v + x \frac{dv}{dx}\). The equation can be solved by variable separable method.

(iii) **Linear Differential Equation**: An equation of the from \(\frac{dy}{dx} + Py = Q\) where \(P\) and \(Q\) are constant or functions of \(x\) only is called a linear differential equation. For finding solution of this type of equations, we find integrating factor \((I.F.) = e^{\int P\,dx}\).

Solution is \(y\ (I.F.) = \int Q\ (I.F.)\,dx + c\)

Similarly, differential equations of the type \(\frac{dx}{dy} + Px = Q\) where \(P\) and \(Q\) are constants or functions of \(y\) only can be solved.

**VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)**

1. Write the order and degree of the following differential equations.

   - (i) \(\frac{dy}{dx} + \cos y = 0\).  
   - (ii) \(\left(\frac{dy}{dx}\right)^2 + 3 \frac{d^2y}{dx^2} = 4\).
   - (iii) \(\frac{d^4y}{dx^4} + \sin x = \left(\frac{d^2y}{dx^2}\right)^5\).
   - (iv) \(\frac{d^5y}{dx^5} + \log \left(\frac{dy}{dx}\right) = 0\).
   - (v) \(\sqrt{1 + \frac{dy}{dx}} = \left(\frac{d^2y}{dx^2}\right)^{1/3}\).
   - (vi) \(1 + \left(\frac{dy}{dx}\right)^2\)^{3/2} = k \frac{d^2y}{dx^2}\).
   - (vii) \(\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = \sin x\).
   - (viii) \(\frac{dy}{dx} + \tan \left(\frac{dy}{dx}\right) = 0\).
2. Write the general solution of following differential equations.

(i) \( \frac{dy}{dx} = x^5 + x^2 - \frac{2}{x} \) 
(ii) \((e^x + e^{-x})\) \( \frac{dy}{dx} = (e^x - e^{-x})dx \)

(iii) \( \frac{dy}{dx} = x^3 + e^x + xe^y \)
(iv) \( \frac{dy}{dx} = 5^{x+y} \).

(v) \( \frac{dy}{dx} = \frac{1 - \cos 2x}{1 + \cos 2y} \). 
(vi) \( \frac{dy}{dx} = \frac{1 - 2y}{3x + 1} \).

3. Write integrating factor of the following differential equations

(i) \( \frac{dy}{dx} + y \cos x = \sin x \)
(ii) \( \frac{dy}{dx} + y \sec^2 x = \sec x + \tan x \)

(iii) \( x^2 \frac{dy}{dx} + y = x^4 \). 
(iv) \( x \frac{dy}{dx} + y \log x = x + y \)

(v) \( x \frac{dy}{dx} - 3y = x^3 \). 
(vi) \( \frac{dy}{dx} + y \tan x = \sec x \)

(vii) \( \frac{dy}{dx} + \frac{1}{1 + x^2} y = \sin x \)

4. Write order of the differential equation of the family of following curves

(i) \( y = Ae^x + Be^{-x} + c \) 
(ii) \( Ay = Bx^2 \)

(iii) \( (x - a)^2 + (y - b)^2 = 9 \) 
(iv) \( Ax + By^2 = Bx^2 - Ay \)

(v) \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \). 
(vi) \( y = a \cos (x + b) \)

(vii) \( y = a + be^{x+c} \)
SHORT ANSWER TYPE QUESTIONS (4 MARKS)

5. (i) Show that \( y = e^{m \sin^{-1} x} \) is a solution of
\[
(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0.
\]

(ii) Show that \( y = \sin(\sin x) \) is a solution of differential equation
\[
\frac{d^2y}{dx^2} + (\tan x) \frac{dy}{dx} + y \cos^2 x = 0.
\]

(iii) Show that \( y = Ax + \frac{B}{x} \) is a solution of
\[
\frac{x^2}{dx^2} + x \frac{dy}{dx} + y = 0.
\]

(iv) Show that \( y = a \cos (\log x) + b \sin (\log x) \) is a solution of
\[
\frac{x^2}{dx^2} + x \frac{dy}{dx} + y = 0.
\]

(v) Verify that \( y = \log \left( x + \sqrt{x^2 + a^2} \right) \) satisfies the differential equation:
\[
(a^2 + x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0.
\]

(vi) Find the differential equation of the family of curves
\( y = e^x (A \cos x + B \sin x) \), where \( A \) and \( B \) are arbitrary constants.

(vii) Find the differential equation of an ellipse with major and minor axes \( 2a \) and \( 2b \) respectively.

(viii) Form the differential equation representing the family of curves \( (y - b)^2 = 4(x - a) \).

6. Solve the following differential equations.

(i) \( \frac{dy}{dx} + y \cot x = \sin 2x \)   (ii) \( x \frac{dy}{dx} + 2y = x^2 \log x \).
(iii) \( \frac{dy}{dx} + \frac{1}{x} \cdot y = \cos x + \frac{\sin x}{x}, \quad x > 0. \)

(iv) \( \cos^3 x \frac{dy}{dx} + \cos x = \sin x. \)

(v) \( ydx + (x - y^3)dy = 0 \)

(vi) \( ye^y dx = \left(y^3 + 2xe^y\right)dy \)

7. Solve each of the following differential equations:

(i) \( y - x \frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right). \)

(ii) \( \cos y \ dx + (1 + 2e^{-x}) \sin y \ dy = 0. \)

(iii) \( x\sqrt{1 - y^2} dx + y\sqrt{1 - x^2} dy = 0. \)

(iv) \( \sqrt{1 - x^2}\sqrt{1 - y^2} \ dy + xy \ dx = 0. \)

(v) \( (xy^2 + x) \ dx + (yx^2 + y) \ dy = 0; \ y(0) = 1. \)

(vi) \( \frac{dy}{dx} = y \sin^3 x \cos^3 x + xy e^x. \)

(vii) \( \tan x \ tan y \ dx + \sec^2 x \ sec^2 y \ dy = 0 \)

8. Solve the following differential equations:

(i) \( x^2 \ y \ dx - (x^3 + y^2) \ dy = 0. \)

(ii) \( x^2 \frac{dy}{dx} = x^2 + xy + y^2. \)

(iii) \( \left(x^2 - y^2\right) dx + 2xy \ dy = 0, \quad y (1) = 1. \)
(iv) \[
\left( y \sin \frac{x}{y} \right) dx = \left( x \sin \frac{x}{y} - y \right) dy.
\]

(v) \[
\frac{dy}{dx} = \frac{y}{x} + \tan \left( \frac{y}{x} \right).
\]

[Hint: \( \frac{x}{y} = v \)]

(vi) \[
\frac{dy}{dx} = -\frac{2xy}{x^2 + y^2}
\]

(vii) \[
\frac{dy}{dx} = e^{x+y} + x^2 e^y.
\]

(viii) \[
\frac{dy}{dx} = \frac{1 - y^2}{1 - x^2}.
\]

(ix) \[
\left( 3xy + y^2 \right) dx + \left( x^2 + xy \right) dy = 0
\]

9. (i) Form the differential equation of the family of circles touching \( y \)-axis at \( (0, 0) \).

(ii) Form the differential equation of family of parabolas having vertex at \( (0, 0) \) and axis along the (i) positive \( y \)-axis (ii) positive \( x \)-axis.

(iii) Form differential equation of family of circles passing through origin and whose centre lie on \( x \)-axis.

(iv) Form the differential equation of the family of circles in the first quadrant and touching the coordinate axes.

10. Show that the differential equation \( \frac{dy}{dx} = \frac{x + 2y}{x - y} \) is homogeneous and solve it.

11. Show that the differential equation :
\[
(x^2 + 2xy - y^2) \, dx + (y^2 + 2xy - x^2) \, dy = 0
\]
is homogeneous and solve it.

12. Solve the following differential equations :

(i) \[
\frac{dy}{dx} - 2y = \cos 3x.
\]

(ii) \[
\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x \quad \text{if} \quad y \left( \frac{\pi}{2} \right) = 1
\]
(iii) \[3e^x \tan y \, dx + \left(1 - e^x\right) \sec^2 y \, dy = 0\]

13. Solve the following differential equations:

(i) \[\left(x^3 + y^3\right) \, dx = (x^2y + x^2y) \, dy.\]

(ii) \[x \, dy - y \, dx = \sqrt{x^2 + y^2} \, dx.\]

(iii) \[y \left(x \cos \left(\frac{y}{x}\right) + y \sin \left(\frac{y}{x}\right)\right) \, dx - x \left(y \sin \left(\frac{y}{x}\right) - x \cos \left(\frac{y}{x}\right)\right) \, dy = 0.\]

(iv) \[x^2 \, dy + y(x + y) \, dx = 0 \text{ given that } y = 1 \text{ when } x = 1.\]

(v) \[\frac{y}{x} e^x - y + x \frac{dy}{dx} = 0 \text{ if } y(e) = 0\]

(vi) \[\left(x^3 - 3xy^2\right) \, dx = (y^3 - 3x^2y) \, dy.\]

(vii) \[\frac{dy}{dx} - \frac{y}{x} + \csc \left(\frac{y}{x}\right) = 0 \text{ given that } y = 0 \text{ when } x = 1\]

14. Solve the following differential equations:

(i) \[\cos^2 x \frac{dy}{dx} = \tan x - y.\]

(ii) \[x \cos x \frac{dy}{dx} + y (x \sin x + \cos x) = 1.\]

(iii) \[\left(1 + e^x\right) \, dx + e^y \left(1 - \frac{x}{y}\right) \, dy = 0.\]

(iv) \[\left(y - \sin x\right) \, dx + \tan x \, dy = 0, \ y(0) = 0.\]
LONG ANSWER TYPE QUESTIONS (6 MARKS EACH)

15. Solve the following differential equations :

(i) \(( x \ dy - y \ dx ) \ y \sin\left( \frac{y}{x} \right) = ( y \ dx + x \ dy ) \ x \cos\left( \frac{y}{x} \right)\)

(ii) \(3e^x \tan y \ dx + (1 - e^x) \sec^2 y \ dy = 0\) given that \(y = \frac{\pi}{4}\), when \(x = 1\).

(iii) \(\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x\) given that \(y(0) = 0\).

**ANSWERS**

1.(i) order = 1, degree = 1  
(ii) order = 2, degree = 1  
(iii) order = 4, degree = 1  
(iv) order = 5, degree is not defined.  
(v) order = 2, degree = 2  
(vi) order = 2, degree = 2  
(vii) order = 3, degree = 2  
(viii) order = 1, degree is not defined

2.(i) \(y = \frac{x^6}{6} + \frac{x^3}{3} - 2 \log |x| + c\)  
(ii) \(y = \log_e \left| e^x + e^{-x} \right| + c\)  
(iii) \(y = \frac{x^4}{4} + e^x + \frac{x^{e+1}}{e+1} + c\).  
(iv) \(5^x + 5^{-y} = c\)  
(v) \(2(y - x) + \sin 2y + \sin 2x = c\).  
(vi) \(2 \log |3x + 1| + 3 \log |1 - 2y| = c\)

3.(i) \(e^{\sin x}\)  
(ii) \(e^{\tan x}\)  
(iii) \(e^{-1/x}\)  
(iv) \(\frac{(\log x)^2}{e^{2}}\)  
(v) \(\frac{1}{x^3}\)  
(vi) \(\sec x\)
(vii) $e^{	an^{-1} x}$

4.(i) 2 
(ii) 1 
(iii) 2 
(iv) 1 
(v) 1 
(vi) 2 
(vii) 2

5.(vi) \[ \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0 \] 
(vii) \[ x \left( \frac{dy}{dx} \right)^2 + xy \frac{d^2 y}{dx^2} = y \frac{dy}{dx} \]
(viii) \[ 2 \frac{d^2 y}{dx^2} + \left( \frac{dy}{dx} \right)^3 = 0 \]

6.(i) \[ y \sin x = \frac{2 \sin^3 x}{3} + c \] 
(ii) \[ y = \frac{x^2 \left( 4 \log_e x - 1 \right)}{16} + \frac{c}{x^2} \]
(iii) \[ y = \sin x + \frac{c}{x}, \quad x > 0 \] 
(iv) \[ y = \tan x - 1 + c e^{-\tan x} \]
(v) \[ xy = \frac{y^4}{4} + c \] 
(vi) \[ x = - y^2 e^y + cy^2 \]

7.(i) \[ cy = (x + 2) (1 - 2y) \] 
(ii) \[ (e^x + 2) \sec y = c \]
(iii) \[ \sqrt{1 - x^2} + \sqrt{1 - y^2} = c \]
(iv) \[ \frac{1}{2} \log \left| \frac{\sqrt{1 - y^2} - 1}{\sqrt{1 - y^2} + 1} \right| = \sqrt{1 - x^2} - \sqrt{1 - y^2} + c \]
(v) \[ (x^2 + 1)(y^2 + 1) = 2 \]
(vi) \[ \log y = -\frac{1}{4} \cos^4 x + \frac{1}{6} \cos^6 x + xe^x - e^x + c \]
\[ = \frac{1}{16} \left[ \cos^3 \frac{2x}{3} - \cos 2x \right] + (x - 1) e^x + c \]

(vii) \[ \log |\tan y| - \frac{\cos 2x}{4} = c \]

8. (i) \[ -\frac{x^3}{3y^3} + \log |y| = c \]
(ii) \[ \tan^{-1} \left( \frac{y}{x} \right) = \log |x| + c \]
(iii) \[ x^2 + y^2 = 2x \]
(iv) \[ y = ce^{\cos(x/y)} \]
(v) \[ \sin \left( \frac{y}{x} \right) = cx \]
(vi) \[ c \left( x^2 - y^2 \right) = y \]

(vii) \[ -e^{-y} = e^x + \frac{x^3}{3} + c \]
(viii) \[ \sin^{-1} y = \sin^{-1} x + c \]
(ix) \[ \left| y^2 + 2xy \right| = \frac{c}{x^2} \]

9. (i) \[ x^2 - y^2 + 2xy \frac{dy}{dx} = 0 \]
(ii) \[ 2y = x \frac{dy}{dx}, \quad y = 2x \frac{dy}{dx} \]
(iii) \[ x^2 - y^2 + 2xy \frac{dy}{dx} = 0 \]
(iv) \[ (x - y)^2 (1 + y')^2 = (x + yy')^2 \]

10. \[ \log \left| x^2 + xy + y^2 \right| = 2\sqrt{3} \tan^{-1} \left( \frac{x + 2y}{\sqrt{3}x} \right) + c \]

11. \[ \frac{x^3}{x^2 + y^2} = \frac{c}{x} (x + y) \]
12. (i) \[ y = \frac{3 \sin 3x}{13} - \frac{2 \cos 3x}{13} + ce^{2x} \]
(ii) \[ y = \frac{2}{3} \sin^2 x + \frac{1}{3} \csc x \]
(iii) \[ \tan y = k \left(1 - e^x\right)^3 \]

13. (i) \[ -y = x \log \left\{ c \left( x - y \right) \right\} \]
(ii) \[ cx^2 = y + \sqrt{x^2 + y^2} \]
(iii) \[ xy \cos \left( \frac{y}{x} \right) = c \]
(iv) \[ 3x^2 y = y + 2x \]
(v) \[ y = -x \log \left( \log |x| \right), \ x \neq 0 \]
(vi) \[ c \left( x^2 + y^2 \right) = \sqrt{x^2 - y^2} \]
(vii) \[ \cos \frac{y}{x} = \log |x| + 1 \]

14. (i) \[ y = \tan x - 1 + ce^{-\tan x} \]
(ii) \[ y = \frac{\sin x}{x} + c \frac{\cos x}{x} \]
(iii) \[ x + ye^y = c \]
(iv) \[ 2y = \sin x \]

15. (i) \[ cxy = \sec \left( \frac{y}{x} \right) \]
(ii) \[ (1 - e^x)^3 \tan y = (1 - e^x)^3 \]
(iii) \[ y = x^2. \]