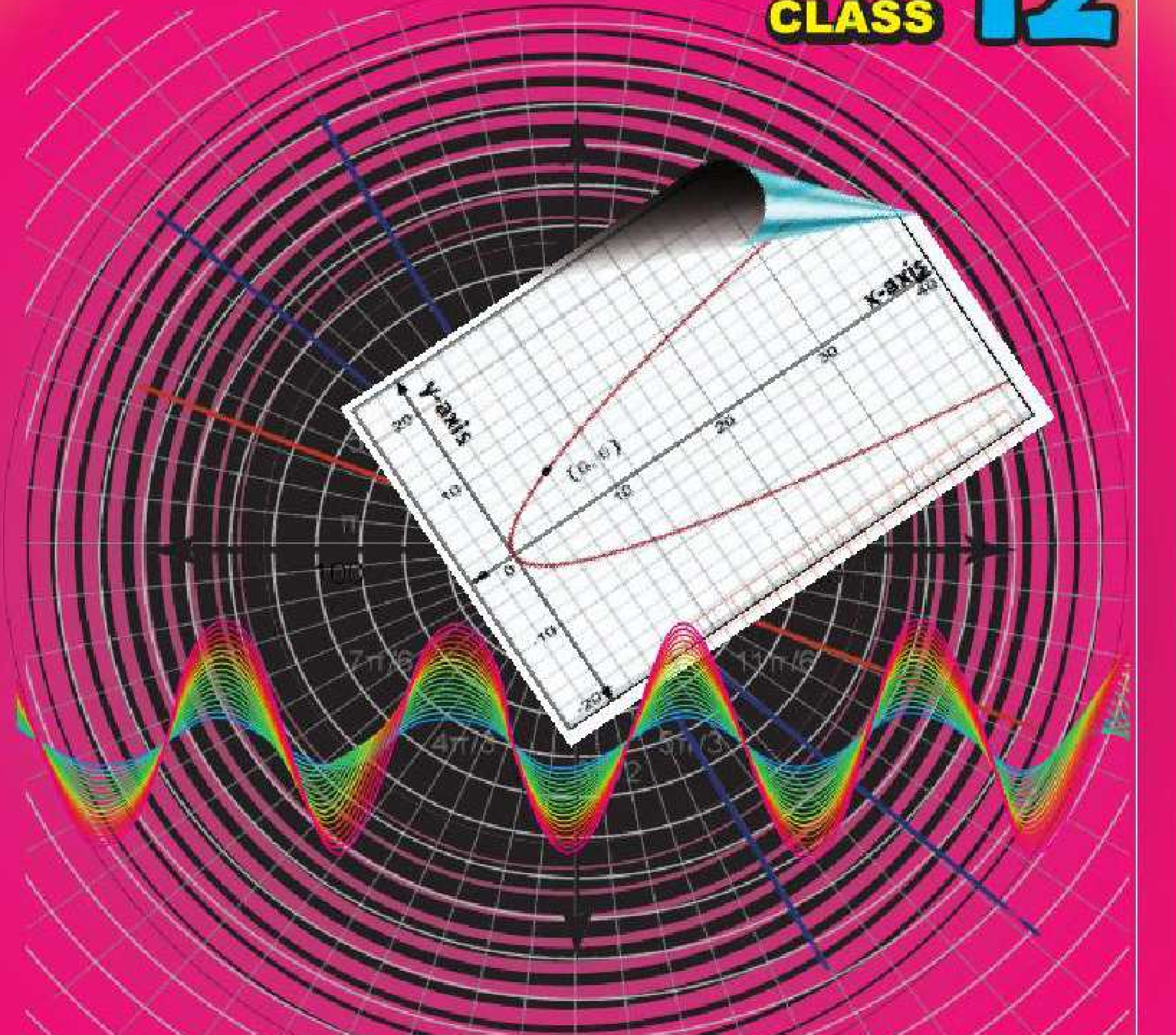
MENTAL MATHS

QUESTION BANK CLASS 12



DIRECTORATE OF EDUCATION GOVT.OF N.C.T.OF DELHI

MENTAL MATHS

CLASS XII

2024-25

DIRECTORATE OF EDUCATION
GOVT. OF NCT OF DELHI

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MESSAGE

The eloquent words of Galileo Galilei resonate: 'The laws of nature are written by the hand of God in the language of mathematics.' In this profound observation, the great astronomer awakened humanity to the paramount importance of mathematics. Within our school education system, mathematics holds a pivotal role, with a dedicated focus on foundational numeracy and literacy.

This year marks a significant milestone, as the project extends its reach to Government-Aided schools and introduces Level IV for classes 11th and 12th as well.

In the competitive arena, where time is of the essence, a strong command over mathematics is indispensable. These skills are not only prized in competitive exams but also wield significant influence in the realms of entrepreneurship and innovation. Mental Maths, with its transformative impact, enhances students' number sense, fosters an understanding of relationships between quantities, and cultivates logical thinking for problem-solving.

The meticulously crafted Mental Maths Question Banks recognize the diverse abilities, needs, and interests of students. As the saying goes, 'Nothing great can be achieved without consistent and persistent hard work'. Heartfelt congratulations to the State Core Team members, District Cordinators and Subject Experts for their silent and steadfast dedication to bring forth these impactful publications.

(Ashok Kumar)

BHUPESH CHAUDHARY, IAS
DIRECTOR (EDUCATION & SPORTS)

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MESSAGE

Beyond mere numbers and equations, Mathematics serves as a foundational language, intricately woven into the fabric of everything from the technology we rely on to the scientific principles shaping our understanding of the cosmos.

Enter Mental Maths – a captivating art of calculation sans paper or tools, a dance of numbers performed within the confines of the mind. It's not just about crunching numbers; it's about empowerment. Mental Maths nurtures the comprehension of place value, fortifies basic operations, and establishes a robust foundation for grappling with more complex mathematical concepts in the future.

Engaging in Mental Maths includes exercising multiple cognitive processes – memory, attention, and critical thinking. Studies reveal that regular Mental Maths exercises contribute to maintaining cognitive reserve, postponing the onset of age-related memory loss, and fending off other cognitive declines. In essence, Mental Maths keeps our minds agile and adaptable, akin to the benefits of physical activity for our bodies. It becomes the catalyst for swift decision-making and adept situational adaptation.

A heartfelt commendation goes to the dedicated State Core Team members and subject experts who meticulously crafted the Mental Maths Question Banks. These resources, tailored for students in Government and Government-Aided Schools of the Directorate of Education are a testament to their sincere efforts and the wise guidance of the Project Director of Mental Maths. It brings me immense pleasure to present this Mental Maths Question Bank to students, encouraging them to weave the magic of Mental Maths into the tapestry of their daily lives.

Bhupesh

(BHUPESH CHAUDHARY)

विकास कालिया
क्षेत्रीय शिक्षा निदेशक
उत्तर एवं मध्य क्षेत्र,
पुरस्कार एवं कल्याण शाखाएँ,
पत्राचार विद्यालय एवं
रा. मुक्त विद्यालयी शिक्षा शाखाएँ
परियोजना निदेशक: मेंटल मैथ्स

सत्यमेव जयते

VIKAS KALIA
Regional Director of Education
Central & North,
Awards & Welfare Branches,
Patrachar Vidyalaya &
NIOS (Branches)
Project Director: Mental Maths

MESSAGE

At the tender age of 16, RPraggnanandhaa, the prodigious talent in Indian chess, sent waves through the global chess community by outsmarting Chess Grandmaster Magnus Carlsen in a lightning-fast game at the Airthings Masters Rapid Chess Tournament. His secret weapon was the remarkable ability for mental calculations. This young genius effortlessly combines his exceptional talent with lightning-quick numerical intuition, fortifying his strategic thinking skills.

At the age of 20, Neelakanta Bhanu Prakash of Hyderabad secures his place as the fastest human calculator on the planet, clinching India's first gold in the Mental Calculation World Championship at the Mind Sports Olympiad in London. Holding an impressive tally of 4 world records and 50 Limca records for speed calculation, his journey is even more remarkable considering a childhood setback. A skull fracture at the age of 5 kept him away from school for a year, but he turned adversity into opportunity, delving into puzzle-solving and mathematics games to hone his cognitive skills.

Mental Mathematics isn't just about acing exams; it's a cognitive superpower that equips the brain to think strategically, break down challenges into manageable steps, and devise creative solutions. This skill transcends academic boundaries, proving invaluable when estimating shopping costs, calculating expenses, or planning a trip. Imagine confidently tallying a shopping bill without reaching for any gadgets.

Recognizing that each student has a unique learning style, Mental Maths Question Banks cater to diverse needs, offering a plethora of materials. Through collaborative efforts, students engage in exhilarating Mental Maths competitions, learning from one another and building self-confidence.

A heartfelt acknowledgment goes to the Mental Maths State Core Team, District and Zonal Coordinators, and HOSs for their unwavering dedication to bringing the Mental Maths superpower to students across all Government and government-aided schools of the Directorate of Education. Gratitude extends to the esteemed Secretary Education and the Director of Education for their guidance and constructive feedback, steering the Mental Maths Project toward continuous improvement.

(VIKAS KALIA)
PROJECT DIRECTOR (MMP)

ACKNOWLEDGEMENT

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STATE LEVEL MENTAL MATH QUIZ COMPETITION RESULT 2023-2024
LEVEL-4

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2	XII	SHIVAM KUMAR	KAUSHAL KISHOR PANDEY	20170069833	RPVV B-1 VASANT KUNJ	1720031	UDAI BIR SINGH
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REGION NORTH (2nd POSITION)

S. No.	CLASS	NAME OF STUDENT	FATHER'S NAME	STUDENT ID	SCHOOL NAME	SCHOOL CODE	NAME OF GUIDE TEACHER
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3	XII	GAUTAM KUMAR THAKUR	MANOJ KUMAR THAKUR	20170207218	RPVV LAJPAT NAGAR	1925334	RAJEEV

REGION CENTRAL (4th POSITION)

S. No.	CLASS	NAME OF STUDENT	FATHER'S NAME	STUDENT ID	SCHOOL NAME	SCHOOL CODE	NAME OF GUIDE TEACHER
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2	XII	MAYANK KUMAR	DINESH KUMAR	20170017235	S. CO-ED V NO. 1, SHAKTI NAGAR	1207037	REETA
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2	XII	PRIYANSHU SHEKHAR	PANKAJ KUMAR JHA	20120017837	SBM SSS SHIVAJI MARG	1516075	ARUN GUPTA
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CONSTITUTION OF INDIA

[1][PART IV A

FUNDAMENTAL DUTIES

Article 51A. Fundamental duties. — It shall be the duty of every citizen of India—

a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;
b) to cherish and follow the noble ideals which inspired our national struggle for freedom;
c) to uphold and protect the sovereignty, unity and integrity of India;
d) to defend the country and render national service when called upon to do so;
e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;
f) to value and preserve the rich heritage of our composite culture;
g) to protect and improve the natural environment including forests, lakes, rivers and wildlife, and to have compassion for living creatures;
h) to develop the scientific temper, humanism and the spirit of inquiry and reform;
i) to safeguard public property and to abjure violence;
j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;]

[2][(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.]

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1. Ins. by the Constitution (Forty-second Amendment) Act, 1976, Sec. 11 (w.e.f. 3-1-1977).
2. Ins. by the Constitution (Eighty-sixth Amendment) Act, 2002, Sec. 4 (w.e.f. 1-4-2010).

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity;

and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty- sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

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1. Subs. by the Constitution (Forty-second Amendment Act,1976, Sec. 2, for "SOVEREIGN DEMOCRATIC REPUBLIC" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment Act,1976, Sec. 2, for "Unity of the Nation" (w.e.f. 3.1.1977)

SCHEDULE OF MENTAL MATHS QUIZ COMPETITIONS

FOR THE YEAR 2024-2025

DIRECTORATE OF EDUCATION

GOVT OF NCT OF DELHI

❖ Practice to students from Question Bank	:	01.04.2024 to 19.10.2024
❖ School Level Quiz Competitions	:	21.10.2024 to 30.10.2024
❖ Cluster Level Quiz Competition	:	14.11.2024 to 20.11.2024
❖ Zonal Level Quiz Competition	:	25.11.2024 to 30.11.2024
❖ District Level Quiz Competition	:	07.12.2024 to 13.12.2024
❖ Regional Level Quiz Competition	:	26.12.2024 to 31.12.2024
❖ State Level Quiz Competition	:	18.01.2025 to 31.01.2025

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CHAPTER 1

RELATIONS AND FUNCTIONS

POINTS TO REMEMBER

1. <u>Cartesian Product of Sets:-</u>

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

Results on Cartesian Products of Sets:

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(iii) $A \times (B - C) = (A \times B) - (A \times C)$.

(iv) If A and B have 'n' elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

(v) If $n(A) = a$ and $n(B) = b$, then

(a) $n(A \times B) = a \times b$ (b) $n(B \times A) = b \times a$

(c) $n(A \times A) = a^2$ (d) $n(B \times B) = b^2$

2. <u>Relation</u>: Let A and B be any two non-empty sets. Any subset of $A \times B$ is said to be a relation from the set A to the set B.

3. <u>Binary Relation on a set</u>: Every subset of $A \times A$ is called a binary relation on A.

I. (i) Identity Relation: $I_A = \{(a, a) : a \in A\}$.

(ii) Universal Relation: $A \times A$ is called the universal relation on A.

II. A relation R on A is said to be:

(i) Reflexive, if $a R a \; \forall \; a \in A$

(ii) Symmetric, if a R b $\Rightarrow$ b R a $\forall$ a, b $\in$ A

(iii) Transitive, if a R b and b R c $\Rightarrow$ a R c $\forall$ a, b & c $\in$ A

(iv) Anti-symmetric, if a R b and b R a $\Rightarrow$ a = b.

III. Equivalence Relation: A relation which is Reflexive, Symmetric and Transitive is called an equivalence relation.

(i) The inverse of an equivalence relation is an equivalence relation.

(ii) The intersection of two equivalence relations is an equivalence relation.

(iii) The union of two symmetric relations is symmetric.

(iv) The union of two transitive relations need not be transitive.

(v) If n (A) = m and n (B) = n, then total no. of relations from A to B is $2^{m.n}$.

(vi) If n (A) = m and n (B) = n, then total no. of relations from B to A is $2^{m.n}$.

(vii) If n (A) = m, then total no. of relations from A to A is 2^{m^2}.

(viii) If n (B) = n, then total no. of relations from B to B is2^{n^2}.

IV. A relation R on a set A is:

(i) Reflexive A $\Leftrightarrow$ $I_A \subseteq$ R.

(ii) Symmetric $\Leftrightarrow$ R^{-1} = R.

(iii) Transitive $\Leftrightarrow$ RoR $\subseteq$ R

4. <u>Functions or Mappings</u>:

Let A and B be two non-empty sets. Then, a rule or a correspondence f which associates to each x$\in$ A, a unique element f (x) $\in$ B, is called a function or a mapping from A to B, and we write, f: A$\rightarrow$ B. f (x) is called the image of x and x is called the pre-image of f (x).

A is called the domain of f and { f (x): x $\in$ A} $\subseteq$ B is called the range of f, where B is the co-domain of f.

5. **Various Types of Functions:**

A function f: A→ B is said to be:

(i) Many-One, if two or more than two elements in A have the same image in B.

(ii) One-One, if distinct elements in A have distinct images in B

i.e., f is one-one, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in A$

(iii) Into, if ∃ at least one element in B which has no pre-image in A.

(iv) Onto, if Range = Co-domain i.e., each element of B has a pre-image in A.

One-one mapping is called Injective; onto mapping is called surjective and a one-one, onto mapping is called bijective.

6. Let f: A→ B and g: B →C, then

gof: A→ C s.t. $(gof)(x) = g(f(x))$

Note I: gof is defined when Range (f) ⊆ Dom (g).

Note II: fog is defined when Range (g) ⊆ Dom (f).

7. Chart to find the domain of various functions:-

	Function	Domain
1.	$\frac{1}{f(x)}$	$D = \{x : f(x) \neq 0\}$
2.	$\frac{f(x)}{g(x)}$	$D = \{x : g(x) \neq 0\}$
3.	$\sqrt{(f(x))}$	$D = \{x : f(x) \geq 0\}$
4.	$\frac{1}{\sqrt{(f(x))}}$	$D = \{x : f(x) > 0\}$
5.	$\log(f(x))$	$D = \{x : f(x) > 0\}$
6.	$\lvert f(x)\rvert, [f(x)], e^{f(x)}$	$D = \text{Domain}(f)$

8. (i) Dom(f + g) = Dom(f) $\cap$ Dom(g).

(ii) Dom (f − g) = Dom (f) $\cap$ Dom (g).

(iii) Dom ($\frac{f}{g}$) = Dom (f) $\cap$ {Dom (g) − { x : g (x) = 0}}

9. To find the range of $f(x)$, firstly put y = f (x) and find value of 'x' in terms of y only, i.e. x = g (y), then find the domain of g (y), this domain (g) = Range (f).

Notes:

(i) As range of all functions cannot be derived by one method. So, students are advised to be a bit careful while finding range.

(ii)If a line parallel to x - axis cuts the graph of f (x) at more than one point, then the function is not One-One.

(iii)If a line parallel to x – axis cuts the graph of f (x) at atmost one point, then the function is One-One.

(iv)If f '(x) remains unchanged in its sign for every point in the domain of the function, then f (x) is always One - One.

QUESTIONS:

1. Let A = {1, 2, 3}, then find the Maximum Number of Reflexive Relation on A?

2. Let B = {m, a, t, h}, then find the total Number of Symmetric Relation on B?

3. Let C = {1, 4, 9}, then find the total Number of Equivalence Relation on C?

4. Let D = {d, o, e}, then write the Smallest Equivalence Relation on D?

5. Let A = {m, e, n, t, a, l}, then find the number of elements in the Smallest Equivalence Relation on A?

6. Let A = {d, o, e}, then find the number of elements in the Largest Equivalence Relation on A?

7. If A = {2, 3, 4} & B = {7, 8}, then find the number of Injective functions from A to B.

8. If A = {1, 2, 3, 4} & B = {5, 6, 7, 8}, then find the number of Injective functions from A to B.

9. If A = {1, 2, 3} & B = {d, o, e}, then find the number of Surjective functions from A to B.

10. If A = {1, 2, 3} & B = {5, 6, 7, 8}, then find the number of Surjective functions from A to B.

11. A relation is defined as $R = \{(a, b): 1 + ab > 0, a, b \in \mathbb{R}\}$. What is the nature of R in terms of Reflexive, Symmetric and Transitive?

12. A relation R on $\mathbb{R}$ is defined as: $aRb \Leftrightarrow |a| \leq b$. What is the nature of R in terms of Reflexive, Symmetric and Transitive?

13. If n (A) = 2 and n (B) = 3, then what are the total number of relations from A to B?

14. What is the nature of the relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} on set A = {1, 2, 3} in terms of Reflexive, Symmetric and Transitive?

15. What is the nature of the relation "less than" on the set of natural numbers in terms of Reflexive, Symmetric and Transitive?

16. For real numbers x and y, we write $x \; R \; y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. What is the nature of the relation R in terms of Reflexive, Symmetric and Transitive?

17. A relation R is defined from {2, 3, 4, 5} to {3, 6, 7, 10} by $x \; R \; y \Leftrightarrow x$ is relatively prime to y. Find the domain of the relation R.

18. What is the nature of the void relation on a set A in terms of Reflexive, Symmetric and Transitive?

19. Let S be the set of all straight lines in a plane. A relation R is defined on S by a R b $\Leftrightarrow$ a ∥ b. What is the nature of R in terms of Reflexive, Symmetric and Transitive?

20. Let A = {1, 2, 3, 4} and let R = {(2, 2), (3, 3), (4, 4), (1, 2)} be a relation on A. What is the nature of R in terms of Reflexive, Symmetric and Transitive?

21. Let f: $\mathbb{N} \rightarrow \mathbb{N}$ such that f (x) = x^2 . What is the nature of f in terms of one-one and onto?

22. Let f: $\mathbb{R} \rightarrow \mathbb{R}$ s.t f (x) = x^2 . What is the nature of f in terms of one-one and onto?

23. Let f : $\mathbb{R} \rightarrow \mathbb{R}$ s.t f (x) = x^3 . What is the nature of f in terms of one-one and onto?

24. Let R^+ be the set of all positive real numbers. A function is defined as

f: $R^+ \rightarrow R^+$ s.t f (x) = e^x. What is the nature of f in terms of one-one and onto?

25. If f: $\mathbb{Q} \rightarrow \mathbb{Q}$ such that $f(x) = 3x + 5$, then find $f^{-1}(x)$.

26. If $f(x) = (x^2 - 1)$ and $g(x) = (3x + 1)$ then find the value of $(g \circ f)(x)$.

27. If $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^1}, x \neq 0$, then what is the value of $f(x)$?

28. Let $f(x) = \sqrt{9 - x^2}$. Find the domain of $f(x)$.

29. What is the domain of $f(x) = \sin^{-1} 2x$?

30. Let $f(x) = \frac{sin^{-1}x}{x}$. What is the domain of $f(x)$?

31. What is the range of $f(x) = a^x$, where $a > 0$?

32. Let f: $\mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{|x|}{x}$, $x \neq 0$, $f(0) = 2$. What is the range of f?

33. What is the range of the function $f(x) = \frac{|x|}{x}$, $x \neq 0$?

34. If f be a function from the set of natural numbers to the set of even natural numbers given by $f(x) = 2x$. What is the nature of f in terms of one-one and onto?

35. Consider the function f: $\mathbb{R} \rightarrow \{0, 1\}$ such that $f(x) = \begin{cases} 1 : x \text{ is rational} \\ 0 : x \text{ is irrational} \end{cases}$.

What is the nature of f in terms of one-one and onto?

36. Let $f(x) = \sqrt{\frac{x-1}{x-4}}$. What is the domain of f?

37. Let $f(x) = \frac{1}{\sqrt{2x-1}} - \sqrt{1 - x^2}$. What is the domain of f?

38. Let $f(x) = \frac{x}{x^1 - 3x + 2}$. What is the domain of f?

39. Let $f : \mathbb{N} \rightarrow \mathbb{N} : f(x) = 2x$. What is the nature of f in terms of one-one and onto?

40. Let f: $R \rightarrow A : f(x) = 3(\sin x) + 4$. Find set A so that f is onto function?

41. Let f: $R \rightarrow A$: $f(n) = n^2 + 2n + 2$. Find set A so that f is onto function?

42. A function is defined as f: $\mathbb{R} \rightarrow A$ s.t $f(x) = \cos x$ such that f(x) is onto. Find set A.

43. Let $f(x) = \cos^{-1}(3x - 1)$. What is the domain of f?

44. If $f(x) = \frac{1-x}{1+x}$, then what is the value of $f[f(\cos 2\theta)]$?

45. Let set A has 3 elements and set B has 4 elements. What are the number of injections that can be defined from A to B?

46. If f: $\mathbb{R} \rightarrow \mathbb{R}$, such that $f(x) = |x|$, then what is the nature of f in terms of one-one and onto?

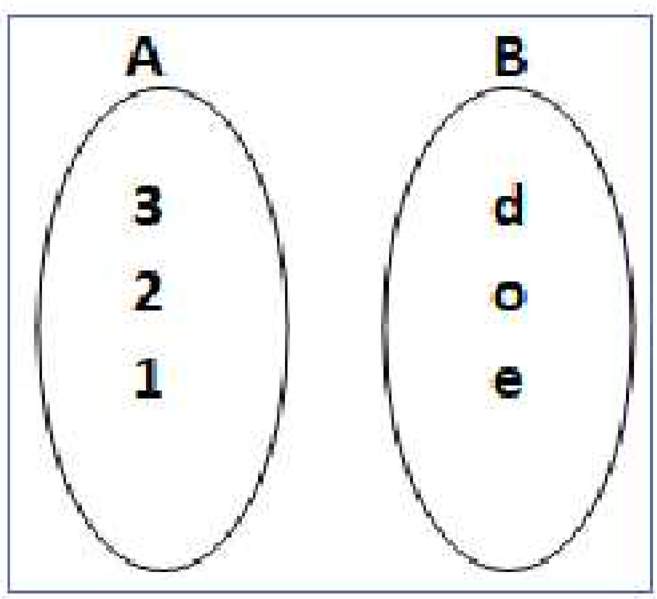
47. What is the domain of $\sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]$?

48. What is the domain of $f(x) = \log|\log x|$?

49. What is the range of the function $f(x) = \frac{x+2}{|x+2|}$?

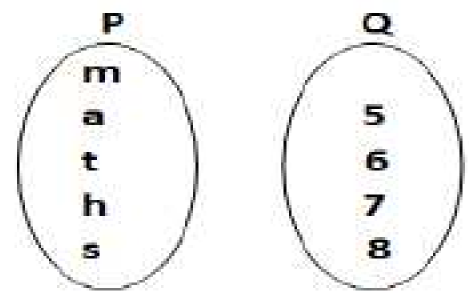
50. What is the range of $f(x) = \cos x - \sin x$?

51. The arrow diagram of a function $f : A \rightarrow B$ is given below.



Find the total number of one-one function from A to B.

52. The arrow diagram of a function f : P → Q is given below.



Find the total number of functions which are one-one & onto both from P to Q.

53. If f: A→[4,∞) such that $f(x) = x^2$ is onto, then what can be the largest set A?

54. If f: A→[4,∞) such that $f(x) = x^2$ is one-one, then what can be the largest set A?

55. If n (A) = 5, then how many bijections can be defined from the set A to A?

ANSWERS

Q. NO.	ANSWER	Q. NO.	ANSWER
1	64	19	Reflexive, Symmetric, Transitive / Equivalence relation
2	1024	20	Transitive neither Reflexive nor Symmetric,
3	5	21	One-one, not onto
4	{(d, d), (o, o), (e, e)}	22	Not one-one, Not onto / Many-one, into
5	5	23	One-one, onto
6	9	24	One-one, not onto
7	0	25	$\frac{x-5}{3}$
8	24	26	$3x^2 - 2$
9	6	27	$x^2 - 2$
10	0	28	[−3, 3]
11	Reflexive, Symmetric, Not transitive	29	[-1/2, 1/2]
12	Not reflexive, not symmetric, Transitive	30	[−1, 1]−{0}
13	64	31	(0, ∞)
14	Reflexive, Transitive but not symmetric	32	{-1, 1, 2}
15	Transitive neither Reflexive nor Symmetric,	33	{-1, 1}
16	Reflexive, Not symmetric, not transitive	34	One-one, onto
17	Domain = {2, 3, 4, 5}	35	Not one-one, onto
18	Symmetric & Transitive but Not reflexive,	36	(−∞ 1]∪(4 ∞)

37	$(\frac{1}{2}, 1]$	47	[1, 9]
38	$\mathbb{R} - \{1, 2\}$	48	$(0, 1) \cup (1, \infty)$
39	One-one, not onto	49	{-1, 1}
40	$[1, 7]$	50	$[-\sqrt{2}, \sqrt{2}]$ (hint: $-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$)
41	$[1, \infty)$	51	6
42	[-1, 1]	52	0
43	$[0, \frac{2}{3}]$	53	$(-\infty, -2] \cup [2, \infty)$
44	cos 2θ	54	$(-\infty, -2]$ or $[2, \infty)$
45	24	55	120
46	Not one-one, not onto		

CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

POINTS TO REMEMBER:

1. (i) $\sin^{-1}x = \theta \Leftrightarrow x = \sin\theta$.

   (ii) $\cos^{-1}x = \theta \Leftrightarrow x = \cos\theta$.

   (iii) $\tan^{-1}x = \theta \Leftrightarrow x = \tan\theta$.

2. Domain & Range:

Functions	Domain (Principle Values)	Range
$\sin^{-1}x$	[-1 , 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	[-1 , 1]	$[0, \pi]$
$\tan^{-1}x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1}x$	$\mathbb{R} - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1}x$	$\mathbb{R} - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

3. (i) $\sin^{-1}(\sin x) = x$, if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

   (ii) $\cos^{-1}(\cos x) = x$, if $0 \le x \le \pi$

   (iii) $\tan^{-1}(\tan x) = x$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

4. (i) $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right), -1 \le x \le 1$

(ii) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right), -1 \le x \le 1$

(iii) $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right), x > 0$

5. (i) $\sin^{-1}(-x) = -\sin^{-1} x, -1 \le x \le 1$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, -1 \le x \le 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

(v) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, |x| \ge 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \ge 1$

6. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, |x| \ge 1$

7. (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, when $x > 0$, $y > 0$ and $xy < 1$

(ii) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, , when $x > 0$, $y > 0$, $xy > 1$

(iii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, when $x > 0$, $y > 0$ and $xy > -1$

8.

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), -1 \le x, y \le 1, x^2 + y^2 \le 1$

(ii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), -1 \le x, y \le 1, x^2 + y^2 \le 1$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-y^2}.\sqrt{1-x^2}\right), -1 \le x, y \le 1, x + y \ge 0$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-y^2}.\sqrt{1-x^2}\right), -1 \le x, y \le 1, x \le y$

9. (i) $\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1), 0 \le x \le 1$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), -1 < x < 1$

$2\tan^{-1}x == \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 \le x < \infty$

$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), -1 \le x \le 1$

10. (i) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3), -\frac{1}{2} \le x \le \frac{1}{2}$

(ii) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), \frac{1}{2} \le x \le 1$

(iii) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

11. (i) For $0 < x < 1$, we have

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \mathrm{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) For $0 < x < 1$, we have

$$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \mathrm{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

(iii) For $x > 0$, we have

$$\tan^{-1}x = \sec^{-1}\sqrt{1+x^2} = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \mathrm{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

(iv) $\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) = \cos^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) = \tan^{-1}\left(\frac{a}{b}\right)$

QUESTIONS:

1. Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

2. Find the principal value of $\cot^{-1}(-\sqrt{3})$.

3. Find the principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

4. Find the principal value of $\cot^{-1}$ (– 1).

5. Find the value of sin ($\cos^{-1}\left(\frac{3}{5}\right)$).

6. Find the value of $\cos\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right)$.

7. Find the value of $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$.

8. Find the value of $\tan^{-1}$ [tan $(\frac{3\pi}{4})$].

9. Find the value of $\sin\left[2\sin^{-1}\left(\frac{4}{5}\right)\right]$.

10. Find the value of $\cos\left[2\tan^{-1}\left(\frac{1}{2}\right)\right]$.

11. Find the value of $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$.

12. Find the value of $\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$.

13. Find the value of $\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$.

14. Convert $\operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$ in terms of $\tan^{-1}x$.

15. If $\tan^{-1} x = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3}\right)$, then what is the value of x ?

16. What is the range of $\tan^{-1}x$?

17. What is the domain of $\cos^{-1} x$?

18. What is the domain of $\sec^{-1} x$?

19. What is the range of $\operatorname{cosec}^{-1} x$?

20. What is the value of $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$?

21. What is the value $\sin^{-1}\{\cos(\sin^{-1}x)\} + \cos^{-1}\{\sin(\cos^{-1}x)\}$?

22. If $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \theta$, then what is the value of $\operatorname{cosec}^{-1}(\sqrt{5})$?

23. Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

24. Find the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$.

25. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$, then what is the value of x ?

26. What is the value of $\tan^{-1}(1/2) + \tan^{-1}(1/3)$?

27. If $\sin^{-1} 1 - \sin^{-1}\frac{4}{5} = \sin^{-1}x$, then what is the value of x?

28. If $\tan^{-1}2$ and $\tan^{-1}3$ are two angles of a triangle, then what is the measure of third angle of the triangle?

29. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x.

30. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then find the value of x.

31. What is the value of $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3}$?

32. What is the value of $\tan^{-1}2 + \tan^{-1}3$?

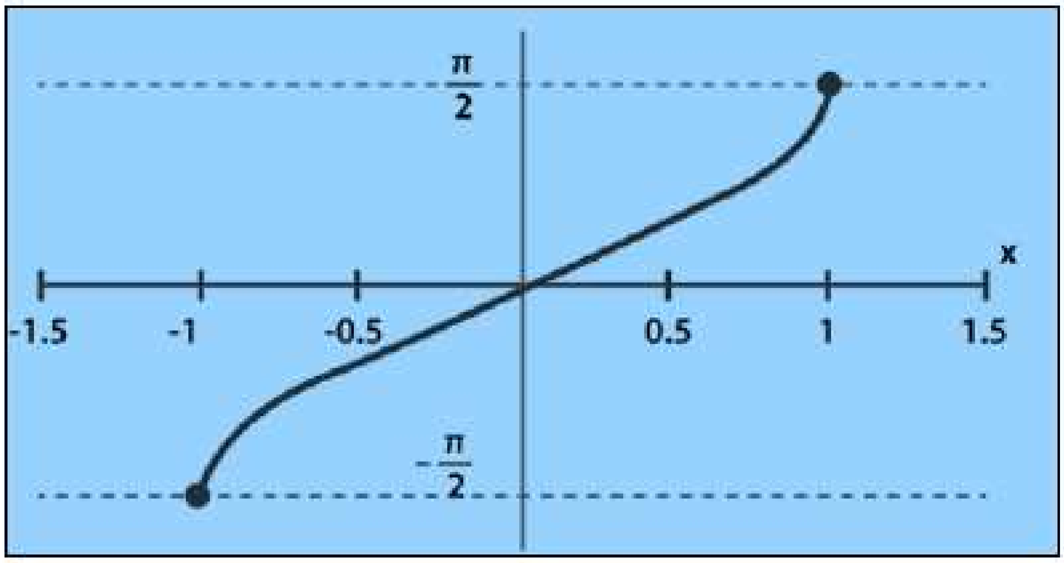
33. If $\sin^{-1}x + \cot^{-1}\frac{1}{2} = \frac{\pi}{2}$, then what is the positive value of x?

34. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then what is the value of x?

35. If $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$, then what is the value of x?

36. Find the value of tan $(\cos^{-1}x)$.

37. The graph of an Inverse Trigonometric Function f(x) is given below:

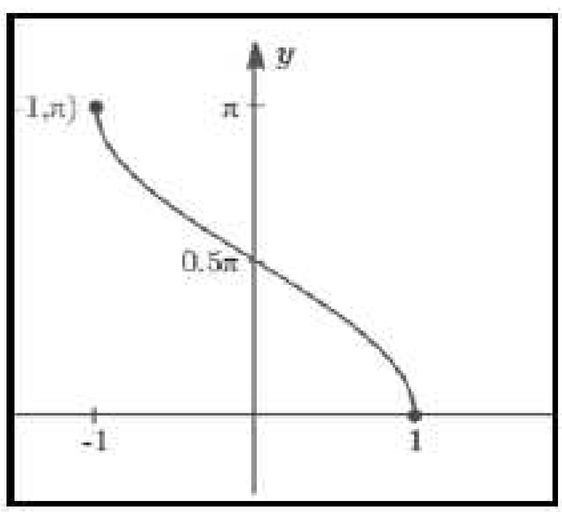


Find the domain of $f(\frac{x}{3})$.

38. What is the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$?

39. What is the value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

40. The graph of an Inverse Trigonometric Function f(x) is given below:



Find the Value of $f(-\frac{1}{2})$.

ANSWERS

Q. NO.	ANSWER	Q. NO.	ANSWER
1	$\frac{2\pi}{3}$	16	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
2	$\frac{5\pi}{6}$	17	[-1 , 1]
3	$-\frac{\pi}{3}$	18	$\mathbb{R}-(-1,1)$
4	$\frac{3\pi}{4}$	19	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
5	$\frac{4}{5}$	20	$\frac{\pi}{3}$
6	0	21	$\frac{\pi}{2}$
7	-1	22	$\frac{\pi}{2}-\theta$
8	$-\frac{\pi}{4}$	23	$-\frac{\pi}{3}$
9	$\frac{24}{25}$	24	1
10	$\frac{3}{5}$	25	$-\frac{1}{\sqrt{3}}$
11	$\frac{2x}{1-x^2}$	26	$\frac{\pi}{4}$
12	$\frac{x}{2}$	27	$\frac{3}{5}$
13	$\frac{x}{2}$	28	$\frac{\pi}{4}$
14	$2\tan^{-1}x$	29	$\frac{1}{5}$
15	$\frac{1}{2}$	30	$\frac{1}{2}$

31	$\frac{\pi}{2}$	36	$\frac{\sqrt{1-x^2}}{x}$
32	$\frac{3\pi}{4}$	37	$[-3\,,3]$
33	$\frac{1}{\sqrt{5}}$	38	$\frac{5\pi}{6}$
34	$\frac{a-b}{1+ab}$	39	$-\frac{\pi}{2}$
35	$\frac{\sqrt{3}}{2}$	40	$\frac{2\pi}{3}$

CHAPTER 3

MATRICES

POINTS TO REMEMBER:

Matrix: A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

The elements in i^{th} row and j^{th} column of a matrix is denoted by a_{ij}.

In general, an m × n matrix has the following rectangular array:

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$A = [a_{ij}]_{m \times n}$, $1 \leq i \leq m$, $1 \leq j \leq n$; $i, j \in \mathbb{N}$

A matrix having m rows and n columns is called a matrix of order m × n or simply m × n matrix (read as m by n matrix).

A matrix having m rows and n columns is called a matrix of order m × n (read as m by n).

2. Diagonal elements of a square matrix: The elements a_{ij} for which i = j are called the diagonal element of the matrix.

3. Comparable matrices: Two matrices A and B are said to be comparable if they are of the same order.

4. Operations on matrices:

(I) Addition and subtraction of matrices: The sum or difference of two matrices is defined only when they are of same order.

If A = $[a_{ij}]_{\mathbf{m}\times\mathbf{n}}$ and B = $[b_{ij}]_{\mathbf{m}\times\mathbf{n}}$ then A+B = $[a_{ij} + b_{ij}]_{\mathbf{m}\times\mathbf{n}}$ and

A − B = $[a_{ij} - b_{ij}]_{\mathbf{m}\times\mathbf{n}}$

(II) Scalar multiplication: Let A= $[a_{ij}]_{\mathbf{m}\times\mathbf{n}}$ and k be any number, then

kA =$[\mathbf{k}a_{ij}]_{\mathbf{m}\times\mathbf{n}}$

(III) Multiplication of matrices: For two matrices A and B, the product AB exist only when number of columns in A = number of rows in B. Otherwise AB does not exist.

Let A = $[a_{ij}]_{\mathbf{m}\times\mathbf{n}}$ and B = $[b_{jk}]_{\mathbf{n}\times\mathbf{p}}$ then C = AB = $[c_{ik}]_{\mathbf{m}\times\mathbf{p}}$ where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

If AB and BA are both defined, then it is not necessary that AB = BA

5. <u>Transpose of a Matrix</u>:

The Matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A, written as $A^{\mathbf{T}}$ or A′.

For any two matrices A and B of suitable orders, we have

(i) (A′)′ = A (ii) (kA)′ = kA′ (where k is any constant)

(iii) (A ± B)′ = A′ ± B′ (iv) (A B)′ = B′ A′

6. <u>Types of matrices</u>:

(I) Diagonal matrix: A square Matrix A= $[a_{ij}]_{\mathbf{m}\times\mathbf{n}}$ in which every non diagonal elements is 0 is called a diagonal matrix.

$$D = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} = \text{diag } [a_{11}, a_{22}, a_{33}, \ldots a_{nn}]$$

(II) Scalar Matrix: A diagonal Matrix in which all diagonal elements are same is called a scalar matrix.

(III) Unit or Identity Matrix: A scalar Matrix with each diagonal element 1, is called a unit Matrix.

We denote a unit matrix of order n by $\mathbf{I_n}$ or I.

(IV) Triangular Matrix: The matrix A= $[a_{ij}]_{n \times n}$ is called:

(i) an upper triangular matrix if $a_{ij} = 0$ when i > j.

(ii) a lower triangular matrix if $a_{ij} = 0$ when i < j.

(V) Symmetric matrix: A square matrix 'A' is said to be symmetric if $(A)' = A$

(VI) Skew-symmetric Matrix: A square matrix 'A' is said to be skew-symmetric if $(A)' = -A$

For any square matrix A, we have

- $A + A'$ is always symmetric.
- $A - A'$ is always skew - symmetric.

QUESTIONS:

1. Find the order of the matrix $\begin{bmatrix} 3 & 5 & \sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$.

2. Find the total number of elements in the matrix $\begin{bmatrix} 3 & 0 \\ 1 & 4 \\ -5 & 6 \end{bmatrix}$.

3. If $A + B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A - B = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$, then find A.

4. Find the additive inverse of the matrix $Y = \begin{bmatrix} 2 & -5 & 0 \\ 4 & 3 & -1 \end{bmatrix}$.

5. If $A = \begin{bmatrix} -2 & 2 \\ -1 & 3 \\ -5 & 1 \end{bmatrix}$, then find 3A.

6. If $\begin{bmatrix} 2x & 4 \end{bmatrix} \begin{bmatrix} x \\ -8 \end{bmatrix} = 0$, then find the value of x.

7. If $A = [a_{ij}] = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix}$, then find the value of $(3a_{12} + 2a_{22})$.

8. Find the order of A^2, if $A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$.

9. Find $(a + b + c + d)$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

10. If $A = [a_{ij}] = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = [b_{ij}] = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find $b_{12} + a_{23}$.

11. If $A = \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}$ and $A + A' = I$, then find the value of α.

12. If $A = \begin{bmatrix} sin\alpha & cos\alpha \\ -cos\alpha & sin\alpha \end{bmatrix}$, then find the value of $A'A$.

13. If $\begin{bmatrix} 2 & 4 \\ 10 & 4 \end{bmatrix} = k\begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix}$, then find the value of k. (k is a scalar)

14. If $A = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ is the sum of symmetric matrix P and skew-symmetric matrix Q, then find the value of Q.

15. For a 3×3 matrix $A = [a_{ij}]_{3 \times 3}$ whose elements are given by $a_{ij} = \frac{i+j}{2}$, find a_{23}.

16. For a 2×2 matrix $B = [b_{ij}]_{2 \times 2}$ whose elements are given by $b_{ij} = \frac{i}{j}$, find b_{12}.

17. If A = $\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$ is the sum of symmetric matrix P and skew-symmetric matrix Q, then find the value of Q.

18. If $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$ + A = $\begin{bmatrix} 9 & -1 & 3 \\ -2 & 2 & 5 \end{bmatrix}$ and matrix A = $[a_{ij}]_{2\times 3}$, then find a_{13}.

19. Find the number of all possible matrices of order 3×3 with each entry 0 or 1.

20. If X = $\begin{bmatrix} -3 & 6 & 9 \\ 15 & 30 & 36 \\ -42 & 45 & 72 \end{bmatrix}$ and $\frac{1}{3}$ X = B where B = $[b_{ij}]_{3\times 3}$, then find b_{22}.

21. If P = $cos\emptyset \begin{bmatrix} cos\emptyset & sin\emptyset \\ -sin\emptyset & cos\emptyset \end{bmatrix} + sin\emptyset \begin{bmatrix} sin\emptyset & -cos\emptyset \\ cos\emptyset & sin\emptyset \end{bmatrix}$, then find matrix P.

22. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, then find the value of x and y.

23. If Q^T = $\begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$ and P = $\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$, then find the sum of all elements of the matrix P − Q.

24. If X = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then find $X^2 - X$.

25. If A = B × I, where B = $\begin{bmatrix} 3 & 11 & 1 \\ 15 & 6 & 1 \\ 12 & 13 & 9 \end{bmatrix}$, I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and A= $[a_{ij}]_{3\times 3}$, then find $a_{23} + a_{13}$.

26. If X + Y = $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and X − Y = $\begin{bmatrix} 0 & 3 \\ 7 & 9 \end{bmatrix}$, then find the value of Y.

27. Find $x + y$, if A = $\begin{bmatrix} 2y-7 & 0 & 0 \\ 0 & x-3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a scalar matrix.

28. Find $x \times y$, if A = $\begin{bmatrix} 2 & 0 & y-x \\ x+y-2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix.

29. If the matrix A = $\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is a skew-symmetric matrix, then find the values of a and b.

30. Let A and B be the matrices of order 3×2 and 2×4 respectively.

Find the order of matrix $(A\,B)'$.

31. Find the number of all possible matrices of order 2×2 with each entry 1, 2 or 3.

32. If $\begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then find the value of $x \times y$.

33. If a matrix, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then find the value of A^3.

34. If a matrix $A = [1 \quad 2 \quad 3]$, then find AA^T.

35. If A = diag [2, 3, -1], B = diag [1, 3, -4], then find the order of A^2B.

36. If A and B are two matrices such that AB = B and BA = A, then find k such that $A^2 + B^2 = k(A + B)$.

37. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find the value of x.

38. If the matrix $A = \begin{bmatrix} 0 & -1 & 3x \\ 1 & y & -5 \\ -6 & 5 & 0 \end{bmatrix}$ is a skew symmetric, then find the value of $6x + y$.

39. If A is a square matrix such that $A^2 = A$, then find the value of $(I + A)^3 - 7A$.

40. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$ and $A = B^2$, then find the value of x.

41. If $A = \begin{bmatrix} 18 & 0 \\ 36 & 6 \end{bmatrix}$ and $kA = \begin{bmatrix} a & 0 \\ 2b & 36 \end{bmatrix}$, then find the value of a and b.

42. If the matrix $A = \begin{bmatrix} -1 & 2 & 3x \\ 2y & 4 & -1 \\ 6 & -1 & 0 \end{bmatrix}$ is symmetric, then find the value of $2x + y$.

43. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then find $A - A^T$.

44. If a matrix A is both symmetric and skew-symmetric, then find the matrix A.

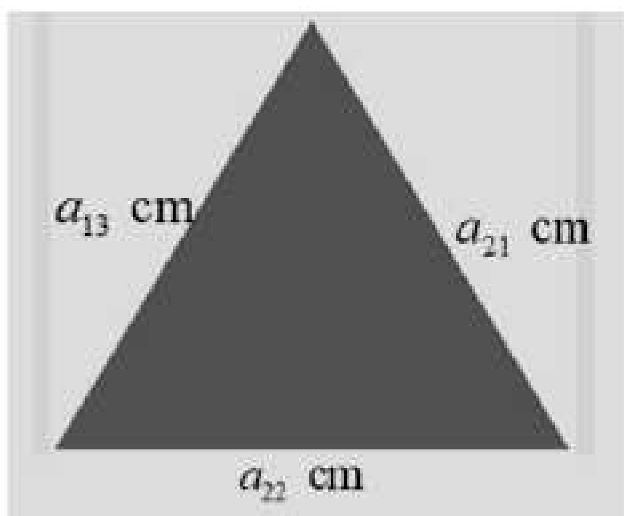
45. If $A = [2 \quad -3 \quad 4]$, $B = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$, $X = [1 \quad 2 \quad 3]$ and $Y = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, then find AB + XY.

46. If the matrix $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$ is the sum of a symmetric matrix B and a skew-symmetric matrix C, then what is the value of $\mathbf{b_{13}}$?

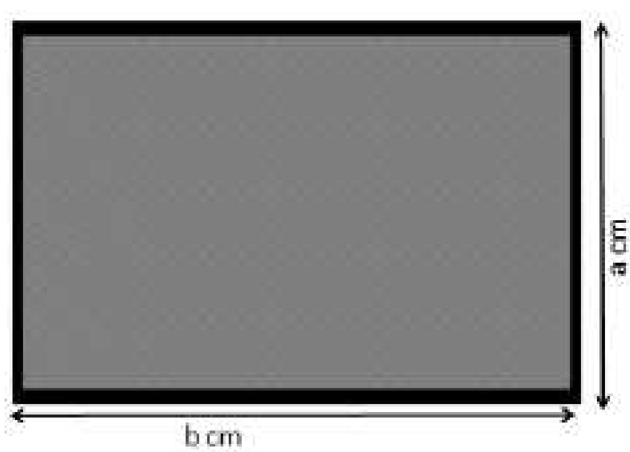
47. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then what is the value of A^{100}?

48. What is the order of the resultant matrix $\begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$?

49. If $A = [a_{ij}]_{3\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then find the Area of Triangle.



50. If $A = [a_{ij}]_{a\times b} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}$, then find the perimeter of Rectangle.



ANSWERS

Q. NO.	ANSWER	Q. NO.	ANSWER
1	2×3	18	4
2	6	19	512 or 2^9
3	$\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$	20	10
4	$\begin{bmatrix}-2 & 5 & 0\\-4 & -3 & 1\end{bmatrix}$	21	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ or I
5	$\begin{bmatrix}-6 & 6\\-3 & 9\\-15 & 3\end{bmatrix}$	22	$x = 3, y = -4$
6	±4	23	0
7	-1	24	$2\,I_{3\times3}$
8	2×2	25	2
9	-10	26	$\begin{bmatrix}1 & -1\\-2 & -2\end{bmatrix}$
10	8	27	17
11	$2n\pi \pm \frac{\pi}{3}, n\epsilon Z$	28	1
12	I or $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	29	$a = -2, b = 3$
13	2	30	4×3
14	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	31	81
15	$\frac{5}{2}$	32	2
16	$\frac{1}{2}$	33	$8\,I_{3\times3}$
17	$\begin{bmatrix}0 & 6\\-6 & 0\end{bmatrix}$	34	[14]

35	3×3	43	$\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$
36	1	44	Null matrix
37	0	45	[28]
38	12	46	7
39	I	47	$2^{99}A$
40	1	48	1×1
41	$a = 108, b = 108$	49	6 $unit^2$
42	5	50	10 units

Chapter 4

DETERMINANTS

POINTS TO REMEMBER:

Determinant:

To every square matrix A =$[a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where a_{ij} = i^{th} row, j^{th} column element of A.

Definition of determinants in terms of function:

If M is the set of square matrices, K is the set of numbers (real or complex) and f : M → K is defined by f (A) = k, where A ∈ M and k ∈ K, then f(A) is called the determinant of A. It is also denoted by | A| or det A or △.

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then determinant of A is written as |A| = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$= det (A)

Remarks:

(i) For matrix A, | A| is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

Properties of Determinants:

Property 1: The value of the determinant remains unchanged if its rows and columns are interchanged.

Property 2: If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

Property 3: If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Property 4: If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

Property 5: If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

Property 6: If each element of a row or column is added with the multiple of another row or element, then the determinant of the matrix remains unchanged.

Key points to remember about determinants:

(1) Let A = $[a_{ij}]$ of order n, then |kA| = k^n |A|

(2) If A and B are square matrix of the same order, then |AB| = |A| |B|.

(3) Let A = $[a_{ij}]$ is a diagonal matrix (lower triangular matrix or upper triangular matrix or scalar matrix) of order n (n ≥ 2), then

$|A| = a_{11} \times a_{22} \times a_{33} \times \dots \times a_{nn}$

Area of a Triangle:

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the expression $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ square units.

Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Remarks:

(i) Since area is a positive quantity, we always take the absolute value of the determinant.

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero and vice versa.

Minors and Co-factors

Definition: Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}

Remark: Minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order $n - 1$.

Definition: Co-factor of an element a_{ij}, denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}.

Note: If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

For example, $\Delta = a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$.

Adjoint of a matrix:

The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by adj A.

Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then adj A = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Theorem 1:

If A be any given square matrix of order n, then A (adj A) = (adj A) A = |A | I, where I is the identity matrix of order n.

Definition: A square matrix A is said to be singular, if |A| = 0.

A square matrix A is said to be non-singular, if |A| ≠ 0

Theorem 2: If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Theorem 3: The determinant of the product of matrices is equal to product of their respective determinants, i.e. |AB| = |A| |B|, where A and B are square matrices of the same order.

Theorem 4: A square matrix A is invertible if and only if A is a non-singular matrix.

$A^{-1} = \frac{adj\, A}{|A|}$ where |A| ≠ 0.

Some important points to remember related to adjoint and inverse of a Matrix.

(1) Let A be a square matrix of order n, then |adj A| = $|A|^{n-1}$.

(2) If A and B are square matrices of same order, then
adj (AB) = (adj B) × (adj A).

(3) If A is an invertible square matrix, then adj (A^T) = $(\text{adj } A)^T$.

(4) Let A be a square matrix of order n, then adj(adj A) = $|A|^{n-2}A$.

(5) Let A be a non-singular square matrix, then $|A^{-1}| = \frac{1}{|A|}$.

(6) If A and B are non-singular square matrices of same order, then
$(AB)^{-1} = (B)^{-1}(A)^{-1}$

(7) If A is an invertible square matrix, then A^T is also invertible and
$(A^T)^{-1} = (A^{-1})^T$

(8) If A is an invertible square matrix, then $A A^{-1} = A^{-1}A = I$ and $(A^{-1})^{-1} = A$.

Solution of system of linear equations using inverse of a matrix:

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be written as, AX = B, i.e.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I: If A is a non-singular matrix, then its inverse exists.

$X = A^{-1} B$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II: If A is a singular matrix, then $|A| = 0$. In this case, we calculate (adj A) B.

If (adj A) B $\neq$ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (adj A) B = O, then system may be either consistent or inconsistent accordingly the system have either infinitely many solutions or no solution.

QUESTIONS:

1. Evaluate $\begin{vmatrix} 2 & 4 \\ -5 & 0 \end{vmatrix}$.

2. Find the values of x for which $\begin{vmatrix} -3 & x \\ x & 2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 4 & 2 \end{vmatrix}$.

3. If X = $\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 79 & 1 & -3 \end{bmatrix}$, then find |X|.

4. Find the sum of minors $\mathbf{M_{12}}$ and $\mathbf{M_{32}}$ of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5. Subtract cofactor C_{13} from C_{32} of the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{bmatrix}$.

6. If A = $\begin{bmatrix} 2x & 6 \\ -1 & 1 \end{bmatrix}$ is a singular matrix, then find x.

7. Evaluate $\begin{vmatrix} \sin 10° & -\cos 10° \\ \sin 80° & \cos 80° \end{vmatrix}$.

8. If C_{ij} is the cofactor of the element a_{ij} of the matrix A = $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$,

   then find the value of $a_{32}C_{32}$.

9. If A= $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, , then find k such that A(adjA) = k I.

10. If A is a square matrix of order 2 and |A| = - 5, then find |adjA|.

11. If A is a square matrix of order 3 and $|A^T|$= - 3, then find $|AA^T|$.

12. If A is a square matrix of order 3 and |A| = 8 , then find |adjA|.

13. If A = $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, then find A(adjA).

14. Find the adjoint of the matrix $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$.

15. If X = $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$, then find $|X^T|$.

16. If A = $2\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, then find $|A^{-1}|$.

17. The area of a triangle with vertices (-3, 0), (3, 0) and (0, k) is 9 square units. Find the value of k.

18. For what value of k, inverse does not exist for matrix $\begin{bmatrix} 1 & 2 \\ k & 6 \end{bmatrix}$.

19. For a non- singular matrix A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $|A^{-1}|$.

20. Given that A is a non-singular matrix of order 3 such that $A^2 = 2A$, then find |2A|.

21. Find x, if $X = \begin{bmatrix} 1 & 2 & x \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ is a singular matrix.

22. For matrix A = $\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix}$, find $|(\text{adj}A)^T|$.

23. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, then find $x + y + z$.

24. If A = $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, X = $\begin{bmatrix} n \\ 1 \end{bmatrix}$, B = $\begin{bmatrix} 8 \\ 11 \end{bmatrix}$ and AX=B, then find n.

25. If A is a square matrix such that A (adjA) = $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then find the value of |adjA|.

26. If A = $\begin{bmatrix} cosx & sinx \\ -sinx & cosx \end{bmatrix}$ and A(adjA) = $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then find the value of k.

27. If A = $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of |adjA|.

28. If A = $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and square matrix B satisfy AB = 8I, then find the value of |B|.

29. If $0 < x < \pi$ and matrix A = $\begin{bmatrix} 4sinx & -1 \\ -3 & sinx \end{bmatrix}$ is singular, then find the value of x.

30. Let A and B be 3 × 3 matrices with |A| = 3 and |B| = 4, then find the value of |2AB|.

31. If A is a square matrix such that $A^3 = I$, then find the value of |A|.

32. Find A^{-1} if A = $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

33. If A = $\begin{bmatrix} a & x \\ y & a \end{bmatrix}$ and $x\,y = 1$, then find $|AA^T|$.

34. Evaluate the determinant $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$.

35. If A is a square matrix of order 3 and A.(adjA) = 10I, then find the value of $\frac{1}{25}|\text{adj}A|$.

36. If a matrix A is such that $4A^3 + 2A^2 + 7A + I = O$, then what is the value of A^{-1}?

37. For non-singular matrices A, B and C of same order, find the value of $(AB^{-1}C)^{-1}$.

38. Find the multiplicative inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$.

39. If A is singular matrix, then find the value of A.(adjA).

40. If a square matrix A is such that $AA^T = I = A^TA$, then what is the value of |A|.

41. If A ≠ 0 and B ≠ 0 are n×n matrices such that AB = O, then what can you say about |A| and |B|?

42. If A is singular matrix, then find the value of $|A^{2024}|$.

43. Find the value of the determinant $\begin{vmatrix} 1 & 5 & \pi \\ log_e e & 5 & \sqrt{5} \\ log_{10}10 & 5 & e \end{vmatrix}$.

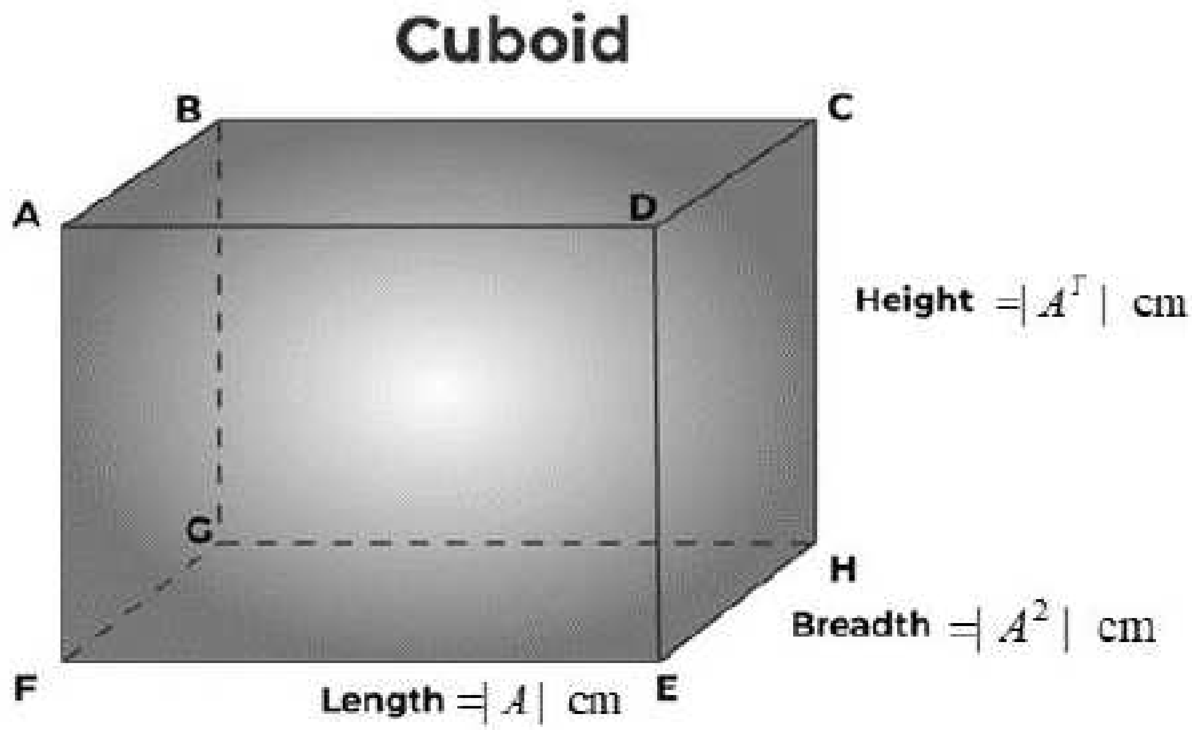
44. If $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 9 & 13 \end{vmatrix}$ and $|B| = \begin{vmatrix} 7 & 20 & 29 \\ 2 & 5 & 7 \\ 3 & 9 & 13 \end{vmatrix}$, then what is the relationship between |A| and |B|?

45. If A and B are square matrices of order 3 such that |A| = −1 and |B| = 3, then what is the value of |3AB|?

46. Find the Mid-point of the line segment PQ, such that $A = \begin{bmatrix} 2 & 0 \\ -5 & 1 \end{bmatrix}$

47. If D = diag[d_1, d_2, d_3], then what is the value of (D^{-1})?

48. If $A = \begin{bmatrix} 1 & 0 \\ 15 & 2 \end{bmatrix}$, then find the Volume of cuboid given Below.



49. What is the value of k for which system equations

$5x + 3y = 3$ and $3x + ky = 8$ has no solution?

50. If |A| = 3 and $A^{-1} = \begin{bmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{bmatrix}$, then what is the value of adjA?

ANSWERS

Q. NO.	ANSWER	Q. NO.	ANSWER
1	20	18	3
2	± 2	19	$-\frac{1}{2}$
3	3	20	64
4	0	21	1
5	10	22	19
6	-3	23	0
7	1	24	2
8	110	25	25
9	1	26	1
10	-5	27	a^6
11	9	28	64
12	64	29	$\frac{\pi}{3}, \frac{2\pi}{3}$
13	I	30	96
14	$\begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}$	31	1
15	10	32	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$
16	$\frac{1}{2}$	33	$(a^2-1)^2$
17	± 3	34	0

35	4	43	0
36	$-4A^2-2A-7I$	44	$\|A\| = \|B\|$
37	$C^{-1}BA^{-1}$	45	-81
38	$\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$	46	(2, 5)
39	Null matrix of order same as A	47	diag $(\frac{1}{d_1}, \frac{1}{d_2}, \frac{1}{d_3})$
40	± 1	48	16 $unit^3$
41	Either $\|A\| = 0$ or $\|B\| = 0$.	49	$\frac{9}{5}$
42	0	50	$\begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$

CHAPTER 5

CONTINUITY AND DIFFERENTIABILITY

POINTS TO REMEMBER:

<u>Limits:</u>

We say $\lim_{x \to c^-} f(x)$ is the expected value of f at x = c given the values of f near x to the left of c. This value is called the left hand limit of f at c.

We say $\lim_{x \to c^+} f(x)$ is the expected value of f at x = c given the values of f near x to the right of c. This value is called the right hand limit of f(x) at c.

If the right and left hand limits coincide, we call that common value as the limit of f(x) at x = c and denote it by $\lim_{x \to c} f(x)$.

<u>Continuity:</u>

Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

More elaborately, if the left hand limit, right hand limit and value of the function at x = c exist and are equal to each other, then f is said to be continuous at x = c.

A real function f is said to be continuous, if it is continuous at every point in the domain of f.

<u>Differentiability:</u>

Suppose f is a real function and 'a' is a point in its domain. The derivative of f at 'a' is defined by

$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided this limit exists. Derivative of f (x) at 'a' is denoted by f′ (a)

The derivative of f at 'x' is defined by $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided this limit exists.

The process of finding derivative of a function is called differentiation.

Theorem: If a function f is differentiable at a point 'c', then it is also continuous at that point.

Corollary: Every differentiable function is continuous whereas a continuous function may or may not be differentiable.

Basic rules for differentiation:

1. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
2. $\frac{d}{dx}(f(x).g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
3. $\frac{d}{dx}[f(g(x))] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}, where\ y = g(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}, g(x) \neq 0$

Basic formulae for differentiation:

1. $\frac{d}{dx}(x^n) = n\,x^{n-1}$
2. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
3. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$
4. $\frac{d}{dx}(e^x) = e^x$
5. $\frac{d}{dx}(a^x) = a^x \log a$
6. $\frac{d}{dx}(\sin x) = \cos x$
7. $\frac{d}{dx}(\cos x) = -\sin x$

8. $\frac{d}{dx}(\tan x) = \sec^2 x$

9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

10. $\frac{d}{dx}(\sec x) = \sec x \tan x$

11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

13. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

15. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$

17. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, $x \in \mathbb{R}$

QUESTIONS:

1. **Determine the value of the constant 'k' so that the function is continuous at x = 3.**

$$f(x) = \begin{cases} kx + 7 \text{ if } x \le 3 \\ x - 2 \text{ if } x > 3 \end{cases}$$

2. **Find the value of 'p' for which function is continuous at x = 2.**

$$f(x) = \begin{cases} px^3 \text{ if } x \le 2 \\ 4 \quad \text{ if } x > 2 \end{cases}$$

3. **Determine the value of the constant 'a' so that the function is continuous at x = 0.**

$$f(x) = \begin{cases} \frac{\sin^3 ax}{x^3} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$$

4. Determine the value of the constant 'k' so that the function is continuous at x = 0.

$$f(x) = \begin{cases} \frac{(1 - \cos 2kx)\cos x}{8x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$$

5. Find the set of points at which the function f(x) = tanx is discontinuous.

6. Find the value of 'b' for which the function is continuous at every point of its domain.

$$f(x) = \begin{cases} 3x + 4 & if\ 0 \leq x \leq 1 \\ 4x^2 + 3bx & if\ 1 < x < 2 \end{cases}$$

7. If the function f(x) is continuous at x = 0, then find the value of k.

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & when\ x \neq 0 \\ k & when\ x = 0 \end{cases}$$

8. If the function f(x) is continuous at x = 0, then find the value of k.

$$f(x) = \begin{cases} \frac{\sin^{-1} x}{2x} & if\ x \neq 0 \\ k & if\ x = 0 \end{cases}$$

9. Determine the value of constant 'k' so that the function f(x) is continuous at x = 0.

$$f(x) = \begin{cases} \frac{kx}{|x|} & when\ x < 0 \\ 3 & when\ x \geq 0 \end{cases}$$

10. Determine the value of constant 'k' so that the function f(x) is continuous at x = 2.

$$f(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2} & when\ x \neq 2 \\ k & when\ x = 2 \end{cases}$$

11. If for the function $f(x) = \lambda x^2 - 4x + 4$ and $f'(2) = 36$, then find the value of λ.

12. Write the value of the derivative of f(x) = |x-1|+|x-3| at x = 2.5

13. If x = cost & y = sin t, then find the value of $(y\frac{dy}{dx} + x)$.

14. Find the derivative of x with respect to $\sqrt{x}$.

15. If $x = at^2$ and y = 2at, then find $\frac{d^2y}{dx^2}$ at t =1.

16. Find $\frac{d}{dx}(\log x^2)$

17. Find $\frac{d}{dx}(\log_{10} x)$

18. Find $\frac{d}{dx}(x^x)$ $at\ x = e.$

19. Find $\frac{d}{dx}\left\{\tan^{-1} \sqrt{\frac{1+\cos x}{1-\cos x}}\right\}$

20. Find $\frac{d}{dx}(\ 2^{-x})$

21. If y =$\sec(\tan^{-1} x)$, then find $\frac{dy}{dx}$.

22. If y = $\frac{\pi}{2} - \cos^{-1}$x, then find $\frac{d^1 y}{dx^1}$.

23. If y = sin3x cos5x, then find $\frac{d^1 y}{dx^1}$.

24. Find $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x)$.

25. If y = $\sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$, then find$\frac{dy}{dx}$.

26. If y = $1 + x + \frac{x^1}{2!} + \frac{x^3}{3!} + \cdots$,then find$\frac{dy}{dx}$.

27. If y = $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ ..., then find$\frac{dy}{dx}$.

28. If y = $x - \frac{x^1}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, then find$\frac{dy}{dx}$.

29. Find$\frac{d}{dx}\cos x°$, where $x° = x$ degree.

30. If y = $\tan^{-1}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$, then find$\frac{dy}{dx}$.

31. Find.$\frac{d}{dx}[\cos^{-1}(4x^3 - 3x)]$.

32. Find the value of $\frac{d}{dx}[\cos^2 x + \sin^2 x + 2024]$.

33. Find the value of $\frac{d}{dx}[\cos^4 x + \sin^4 x + 2025]$

34. Find the value of $\frac{d}{dx}[1 - 2\cos^2 x]$

35. Find the derivative of f(x) = | x |^3 at $x = -1$.

36. Find the points of continuity of f(x)= | x -1| +| x +1|.

37. Find the value of $\frac{d}{dx}\left[\frac{1-\tan^1 x}{1+\tan^1 x}\right]$

38. If $y = e^{2x}$, then find the value of k such that $\frac{d^2y}{dx^2} = ky$.

39. If y = sin3x, then find the value of k such that $\frac{d^2y}{dx^2} = ky$.

40. If $f(x) = |x - 2|$, then at how many points f(x) is not differentiable.

41. If $f(x) = |\sin x|$, then find the points of differentiability.

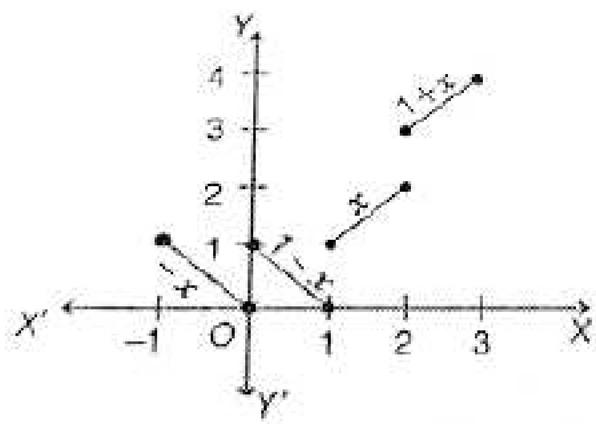
42. If $f(x) = \sin|x|$, then find the points of differentiability.

43. If $f(x) = \cos x$, then find the points of differentiability.

44. If $f(x) = |\log x|$, then find the points of non-differentiability.

45. The graph of a function y = f(x) defined on [-1, 3] is given below.

Find the number of integral points where f(x) is differentiable.



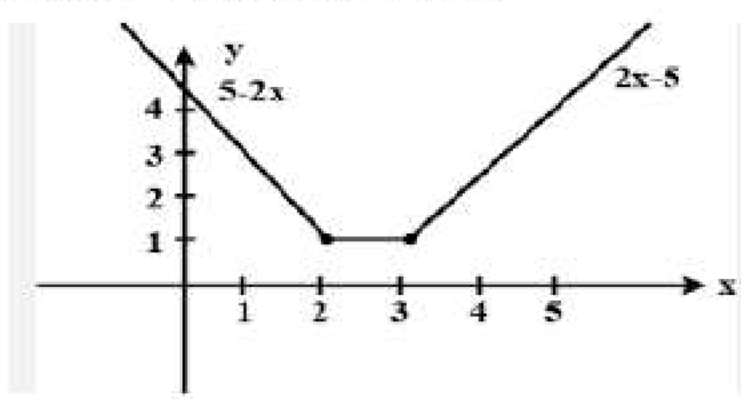
46. Find the derivative of $\cos^3 x$ with respect to $\sin^3 x$.

47. Find the points where $f(x) = |x| + |x - 1|$ is continuous, but not differentiable.

48. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots\infty}}}$, then find $\frac{dy}{dx}$.

49. If $f(x) = [x]$, where $1 < x < 10$, then find the points where f(x) is not continuous.

50. The graph of a function y = f(x) is given below.



Find the value of $f(2.0505) + f'(2.0505)$.

ANSWERS

Q. NO.	ANSWER	Q. NO.	ANSWER
1	-2	19	$-\frac{1}{2}$
2	$\frac{1}{2}$	20	$-2^{-x}\log 2$
3	1	21	$\frac{x}{\sqrt{1+x^2}}$
4	±2	22	$\frac{x}{(1-x^2)^{\frac{3}{2}}}$
5	$\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$	23	2 sin 2x −32 sin 8x
6	1	24	0
7	2	25	0
8	$\frac{1}{2}$	26	e^x
9	-3	27	cos x
10	7	28	$\frac{1}{x+1}$
11	10	29	$-\frac{\pi}{180°}(\sin x°)$
12	0	30	-1
13	0	31	$-\frac{3}{\sqrt{1-x^2}}$
14	$2\sqrt{x}$	32	0
15	$\frac{-1}{2a}$	33	$-sin4x$
16	$\frac{2}{x}$	34	2Sin2x
17	$\frac{1}{x(log_e 10)}$	35	3
18	$2e^e$	36	$\mathbb{R}$

37	-2sin2x	44	x = 1
38	4	45	0
39	-9	46	– cot x
40	1	47	$\{0, 1\}$
41	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$	48	$\frac{\cos x}{2y-1}$
42	$\mathbb{R} - \{0\}$	49	8
43	$\mathbb{R}$	50	1

CHAPTER 6

APPLICATION OF DERIVATIVES

Points to remember:

Rate of change: Whenever one quantity y varies with respect to another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $f'(x_1)$ or $\frac{dy}{dx}$ at $x = x_1$, represents the rate of change of y with respect to x at $x = x_1$.

Increasing and decreasing functions:

Let I be an interval in the domain of a real valued function f. Then f is said to be

(i) Increasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in$ I.

(ii) Strictly increasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) Decreasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(iv) Strictly decreasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

(v) Constant on I, if $f(x) = c$ for all $x \in$ I, where c is a constant.

Derivative test:

Let f be continuous on [a, b] and differentiable on (a,b), then

(a) f is increasing in [a,b], if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in [a,b], if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in [a,b] if $f'(x) = 0$ for each $x \in (a, b)$

Maxima and Minima:

Let f be a function defined on an interval I. Then

(a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in$ I. The value f(c) is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$. The value f (c), in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I. The value f (c), in this case, is called an extreme value of f in I and the point c is called an extreme point.

By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Every continuous function on a closed interval has a maximum and a minimum value.

Theorem: Let f be a function defined on an open interval I. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c , where $c \in I$.

Critical points: A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

Theorem (First Derivative Test): Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I, then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e. if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of local maxima.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflexion.

If c is a point of local maxima of f, then f(c) is a local maximum value of f. Similarly, if c is a point of local minima of f, then f(c) is a local minimum value of f.

Theorem (Second Derivative Test): Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c, then

(i) x = c is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$.
The value f (c) is local maximum value of f.

(ii) x = c is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.
In this case, f (c) is local minimum value of f.

(iii) The test fails, if $f'(c) = 0$ and $f''(c) = 0$.
In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Theorem: Let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem: Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$, if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$, if f attains its absolute minimum value at c.

QUESTIONS:

1. Find the intervals in which function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing.
2. Find the intervals in which function $f(x) = x^9 + 3x^7 + 6$ is increasing.
3. Find the intervals in which function $f(x) = x^3 + x$ is decreasing.
4. Find the minimum value of $(x^2 + \frac{250}{x})$ where, $x > 0$.
5. If $x > 0$ and $x + y = 18$, then find the maximum value of xy.
6. Find the least value of $f(x) = e^x + e^{-x}$.
7. Find the maximum area of rectangle of perimeter 160 cm.
8. The sum of two positive number is 60. If the sum of their squares is minimum, then find the numbers.

9. If $f(x) = x^2 e^{-x}$, then find the intervals in which function is increasing.

10. Find the minimum value of $f(x) = |x+2|$.

11. Find the point of maxima of $f(x) = \cos x$.

12. Find the extreme point(s) of $f(x) = \tan x$.

13. Find the intervals in which the function $f(x) = x^3 - 6x^2 + 9x + 2$ is monotonically decreasing.

14. Find the intervals in which $f(x) = \sin x - \cos x$ is decreasing, where $0 < x < 2\pi$.

15. Let $f(x) = x^3$ defined on [-2, 2], then find the absolute maximum value of f(x).

16. If x is real, then find the minimum value of $f(x) = x^2 - 8x + 17$.

17. Find the intervals in which $f(x) = [x(x-3)]^2$ increases for all value of x.

18. For what value(s) of x the function f(x) = cosecx has a maxima in the interval $(-\pi, \mathbf{0})$?

19. Find the intervals in which $f(x) = \sin^2 x$ defined on $(0, \pi)$ is increasing.

20. Find the minimum value of the function $f(x) = \sin 3x$.

21. Find the maximum value of the function $f(x) = -|x-1| - 3 \ \forall x \in \mathbb{R}$.

22. Find the minimum value of the function $(x-a)(x-b)$.

23. Find the maximum value of $f(x) = \frac{\log x}{x}$.

24. Find the minimum value of $f(x) = \sin x + \cos x$.

25. If the function $f(x) = a^x$ is strictly increasing, then find the interval in which 'a' lies.

26. In which interval the function $f(x) = \tan x - x$ is a decreasing function?

27. In which interval the function $f(x) = |x|$ is strictly decreasing?

28. Find the interval in which $f(x) = x - \sin x$ is increasing.

29. Find the values of 'a' for which $f(x) = x^3 - ax$ is an increasing function on $\mathbb{R}$.

30. Find the value(s) of 'b' for which the function $f(x) = \sin x - bx + c$ is a decreasing function on $\mathbb{R}$.

31. At what points, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum.

32. Find the value of 'a' for which the function $f(x) = x^4 - 62x^2 + ax + 3$ attains its maximum value at x = 1 in the interval [0, 3].

33. What is the least value of function $f(x) = x^2 + x + 1$?

34. If x lies in the interval [0, 1], then find the least value of the function $f(x) = x^2 + x + 4$.

35. Find the maximum value of sinxcosx.

36. What is the rate of change of the circumference of a circle with respect to radius r when r = 4 m?

37. The total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 4x^2 + 26x + 1$. Find the marginal revenue when as x = 10.

38. Find the Minimum value of the function f(x), where

$$f(x) = \frac{2}{x} + \frac{x}{2}, x > 0$$

39. If $C(x) = 100 + 300x - 3x^2$, Find the value of 'x' for which C(x) is decreasing.

40. Find the absolute maximum value of $y = x^3 - 3x + 2$ in $0 \le x \le 2$.

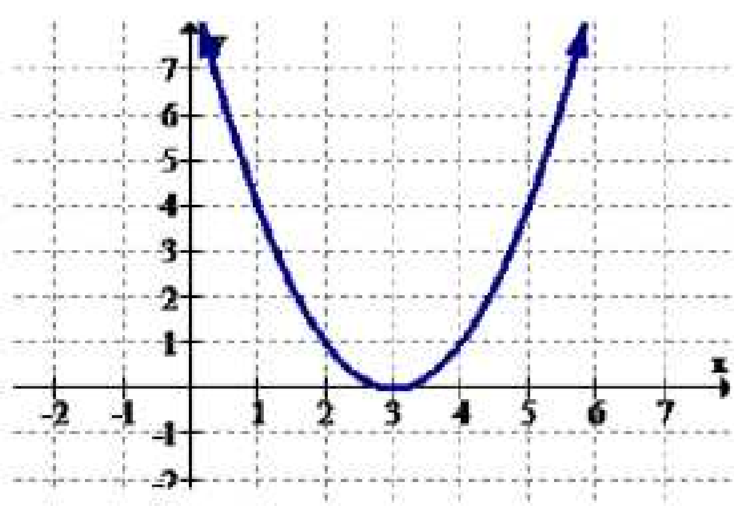
41. Find the point(s) on the curve $y = 3x^2 + 1$, at which y-coordinate is changing six times as fast as x-coordinate.

42. The rate of change of y with respect to x is 3. Find the equation of line when x = 0, y = 2.

43. The distance covered by a particle (in metres) in t seconds is given by $x = 5 + 7t - 2t^2$. What will be its velocity after 1 second?

44. Find the least value of the function $f(x) = 2\cos x + x$ in $[0, \pi/2]$.

45. The Graph of a function f(x) is shown below, Find the interval where the function f(x) is Increasing.



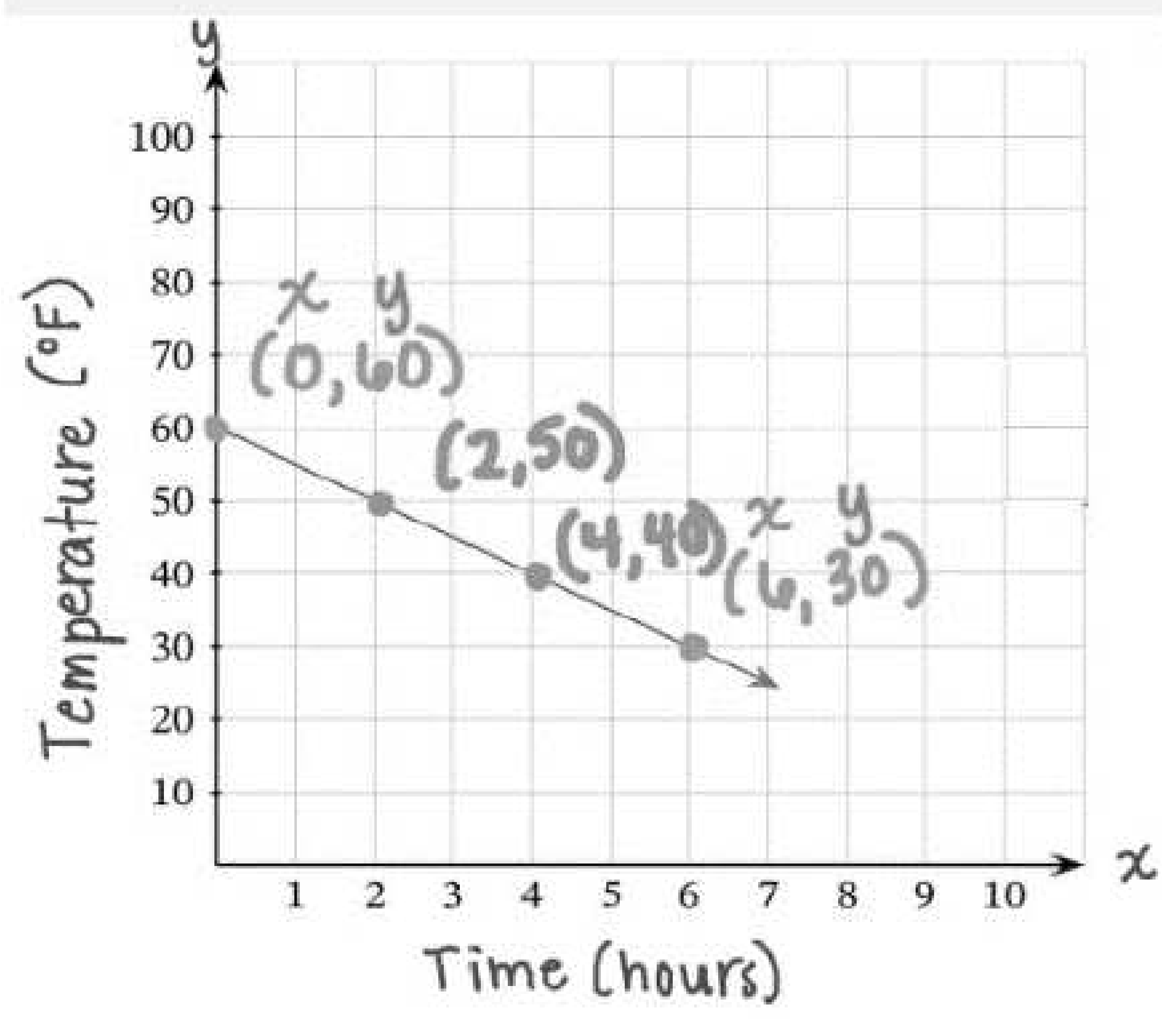
46. Find the value of x for which $f(x) = - x^2 + 2024$ is strictly increasing function.

47. Find the Maximum value of $f(x) = - x^2 + 2x$.

48. Find an angle θ, which increases twice as fast as its cosine where $0 < \theta < 2\pi$.

49. If the function $f(x) = x^3 - 12x^2 + kx - 8$ attains its maximum value at $x = 1$ in the interval [0, 3], then find the value of k.

50. The graph given below is showing the variation of Temperature with time. Find the rate of change of temperature with time.



ANSWERS

Q. NO.	ANSWER	Q. NO.	ANSWER
1	$(-\infty, 2) \cup (3, \infty)$	26	Ø
2	$\mathbb{R}$	27	$(-\infty, 0)$
3	Ø	28	$\mathbb{R}$
4	75	29	$a \leq 0$
5	81	30	$b > 1$
6	2	31	$(1, -16)$
7	1600 cm^2	32	120
8	30, 30	33	$\frac{3}{4}$
9	$(0, 2)$	34	4
10	0	35	$\frac{1}{2}$
11	$x = 2n\pi, \forall n \in \mathbb{Z}$	36	2π m/sec
12	Ø	37	₹ 106
13	$(1, 3)$	38	2
14	$(\frac{3\pi}{4}, \frac{7\pi}{4})$	39	$x > 50$
15	8	40	4
16	1	41	$(1, 4)$
17	$(0, \frac{3}{2}) \cup (3, \infty)$	42	$y = 3x + 2$
18	$-\frac{\pi}{2}$	43	3m/sec
19	$(0, \frac{\pi}{2})$	44	$\frac{\pi}{2}$
20	-1	45	$[3, \infty)$
21	-3	46	$(-\infty, 0)$
22	$-\frac{(a-b)^2}{4}$	47	1
23	$\frac{1}{e}$	48	$\frac{7\pi}{6}, \frac{11\pi}{6}$
24	$-\sqrt{2}$	49	21
25	$(1, \infty)$	50	$-5\ {}^0F/hr$

CHAPTER 7

INTEGRALS

POINTS TO REMEMBER:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$

2. $\int \frac{1}{x} dx = \log|x| + C$

3. $\int e^x dx = e^x + C$

4. $\int a^x dx = \frac{a^x}{\log a} + C$

5. $\int \sin x \, dx = -\cos x + C$

6. $\int \cos x \, dx = \sin x + C$

7. $\int sec^2 x \, dx = \tan x + C$

8. $\int cosec^2 x \, dx = -\cot x + C$

9. $\int \sec x \tan x \, dx = \sec x + C$

10. $\int cosec \, x \cot x \, dx = -cosec \, x + C$

11. $\int \tan x \, dx = \log|\sec x| + C = -\log|\cos x| + C$

12. $\int \cot x \, dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

13. $\int \sec x \, dx = \log|\sec x + \tan x| + C = \log tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$

14. $\int cosec \, x dx = \log|cosec \, x - \cot x| + C = \log|\tan\frac{x}{2}| + C$

15. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log|\frac{x-a}{x+a}| + C$

16. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log|\frac{a+x}{a-x}| + C$

17. $\int \frac{1}{a^1+x^1} dx = \frac{1}{a} tan^{-1} \frac{x}{a} + C$

18. $\int \frac{1}{\sqrt{a^1-x^1}} dx = sin^{-1} \frac{x}{a} + C$

19. $\int \frac{1}{\sqrt{x^1-a^1}} dx = \log|x + \sqrt{x^2 - a^2}| + C$

20. $\int \frac{1}{\sqrt{x^1+a^1}} dx = \log|x + \sqrt{x^2 + a^2}| + C$

21. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^1}{2} sin^{-1} \frac{x}{a} + C$

22. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^1}{2} \log|x + \sqrt{x^2 - a^2}| + C$

23. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^1}{2} \log|x + \sqrt{a^2 + x^2}| + C$

24. Integration by parts: $\int u.v dx = u.\int v dx - \int \left\{\frac{du}{dx}.\int v dx\right\} dx + C$

25. $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

26. $\int e^{ax} \sin(bx + c)\, dx = \frac{e^{ax}}{a^1+b^1}[\mathrm{a}.\sin(bx + c) - b.\cos(bx + c)] + C$

27. $\int \frac{p.\sin x + q.\cos x}{a.\sin x + b.\cos x} dx = Ax + B.\log|a.\sin x + b.\cos x| + C$ where,

$A = \frac{ap+bq}{a^1+b^1}$ & $B = \frac{aq-bp}{a^1+b^1}$

28. $\int e^{mx}.\sin(nx)\, dx = \frac{e^{mx}}{m^1+n^1}[m.\sin(nx) - n.\cos(nx)] + C$

29. $\int e^{mx}.\cos(nx)\, dx = \frac{e^{mx}}{m^1+n^1}[m.\sin(nx) + n.\cos(nx)] + C$

30. $\int \frac{1}{a^1 cos^1 x + b^1 sin^1 x} dx = \frac{1}{ab} tan^{-1}\left(\frac{b}{a}\tan x\right) + C$

32. $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$, where F(x) $=\int f(x)dx$

33. $\int_a^b f(x)dx = \int_a^b f(t)dt$

34. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

35. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

36. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

37. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

38. $\int_{-a}^a f(x)dx = \int_0^a [f(x) + f(-x)]\, dx$

39. $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a-x)]\, dx$

40. If f(x) is a periodic function with period 'T', then $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

41. Walli's Formula:-

$$\int_0^{\frac{\pi}{2}} sin^n\, x dx = \int_b^{\frac{\pi}{2}} cos^n\, x dx = \begin{cases} \frac{(n-1)(n-3)(n-5)\dots 1}{n.(n-2)(n-4)\dots 2}.\frac{\pi}{2} & if\ n\ is\ even \\ \frac{(n-1)(n-3)(n-5)\dots 2}{n.(n-2)(n-4)\dots 1}.1 & if\ n\ is\ odd \end{cases}$$

42. $\int_a^b |f(x)|\, dx$, limit of this integral will split at all those points for which f(x)=0 and a< f(x)< b

43. $\int_0^{\frac{\pi}{2}} \log|\sin x|\, dx = \int_0^{\frac{\pi}{2}} \log|\cos x|\, dx = -\frac{\pi}{2}\log 2$

44. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx = \frac{1}{2}(b-a)$

45. $\int_a^b \frac{1}{a^2 cos^2 x + b^2 sin^2 x} dx = \frac{\pi}{2ab}$

46. $\int_0^{\frac{\pi}{2}} \frac{a\sin x + b.\cos x}{\sin x + \cos x} dx = \frac{\pi}{4}(a+b)$

47. $\int_0^{\frac{\pi}{2}} \frac{a.\tan x + b.\cot x}{\tan x + \cot x} dx = \frac{\pi}{4}(a+b)$

48. $\int_0^{\frac{\pi}{2}} \frac{a.\operatorname{cosec} x + b.\sec x}{\operatorname{cosec} x + \sec x} dx = \frac{\pi}{4}(a+b)$49.

$$\int_0^{\pi} \sin ax . \cos bx\, dx = \begin{cases} \frac{2a}{a^2-b^2} & : if\ a-b\ is\ odd \\ 0 & : if\ a-b\ is\ even \end{cases}$$

QUESTIONS:

1. Find $\int \frac{x^4+5x^3}{x^3} dx$
2. Find $\int \frac{x^3-3x^2+3x-1}{x-1} dx$
3. Find $\int \frac{cosec\, x}{\sec x} dx$
4. Find $\int \frac{(x-1)(x-\log x)}{x} dx$
5. Find $\int \cot 2x \,.\log \sin 2x\, dx$
6. Find $\int \log 3x\, dx$
7. Find $\int cosec\, x\,.\log|cosec\, x - \cot x|\, dx$
8. Find $\int e^{sin^2 x} \sin x \cos x\, dx$
9. Find $\int \frac{x^4+1}{x^2+1} dx$
10. Find $\int \frac{1-\cot x}{1+\cot x} dx$
11. Find $\int \sqrt{ax+b} dx$
12. Find $\int \frac{1}{\sqrt{x}+x} dx$
13. Find $\int \sqrt{\frac{1-x}{1+x}} dx$
14. Find $\int \frac{1}{1-\cos x} dx$
15. Find $\int \frac{\cos 2x}{\cos x} dx$
16. Find $\int cos^2 nx\, dx$
17. Find $\int \sin 3x \,.\, \sin 2x\, dx$
18. Find $\int \frac{\sin x}{\sin(x+a)} dx$
19. Find $\int (4sin^3 x - 3 \sin x) dx$
20. Find $\int 4cos^3(2x+1) - 3\cos(2x+1)\, dx$
21. Find $\int \frac{f'(x)}{f(x) \log f(x)} dx$
22. Find $\int \frac{f'(x)}{\sqrt{f(x)}} dx$

23. Find $\int \frac{\cot x}{\log \sin x} dx$

24. Find $\int \frac{e^{tan^{-1}\sqrt{x}}}{\sqrt{x}+x\sqrt{x}} dx$

25. Find $\int \frac{2^x}{\sqrt{1-4^x}} dx$

26. Find $\int x \ sin^3 x^2 . \cos x^2 \ dx$

27. Find $\int sinx.cosx.tanx.cotx.secx.cosecx.dx$

28. Find $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

29. Find $\int \sqrt{1+\sin 2x}\, dx$

30. Find $\int \frac{\sin x}{(1+\cos x)(2+\cos x)} dx$

31. Find $\int \frac{1}{2x^2+4x+4} dx$

32. Find $\int \frac{1}{(x+1)(x+2)} dx$

33. Find $\int e^x \left[tan^{-1}x + \frac{1}{x^2+1}\right] dx$

34. Find $\int e^x . \frac{x-3}{(x-1)^3} dx$

35. Find $\int e^x \sec x\,(1+\tan x)\, dx$

36. Find $\int tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx$

37. Find $\int cot^{-1}\left(\frac{\sin 4x}{1-\cos 4x}\right) dx$

38. Find $\int \frac{sin^6 x + cos^6 x}{sin^6 x \ cos^2 x} dx$

39. Find $\int tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right) dx$

40. Find $\int \frac{\sec x}{\sec x - \tan x} dx$

41. Find $\int \frac{1}{\sin(x+a).\cos(x+b)} dx$

42. Find $\int \frac{\sin x}{\sqrt{sin^2 x - sin^2 \alpha}} dx$

43. Find $\int a^x e^x dx$

44. Find $\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx$

45. Find $\int_1^e \frac{(\log x)^2}{x} dx$

46. Find $\int_0^1 \frac{x^3}{1+x^8} dx$

47. Find $\int_{\sqrt{3}}^{\sqrt{8}} x.\sqrt{1+x^2}dx$

48. Find $\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x} dx$

49. Find f(0), if $\int e^x(-\sin x+\cos x)dx = f(x)+c$

50. Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

51. Let [x] denote the greatest integer function less than or equal to x, then find $\int_{-1}^{1}[x]dx$.

52. Let [x] denote the greatest integer function less than or equal to x, then find $\int_0^2 x[x]dx$.

53. Find $\int_0^{\frac{\pi}{2}} \frac{f(x)}{f(x)+f\left(\frac{\pi}{2}-x\right)} dx$

54. Find $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$

55. Find $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}} dx$

56. Find $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x}{\cos x+\sin x} dx$

57. Find $\int_{-\pi}^{\pi} x^3 \cos^3 x \, dx$

58. Find $\int_{-a}^{a} x\sqrt{a^2-x^2}dx$

59. Find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x| dx$

60. Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin x \cos x} dx$

61. Find $\int_0^1 x.e^{x^2} dx$

62. Find $\int_a^b \frac{\log x}{x} dx$

63. Find $\int_{-2}^{2} (x+\sin x)^9 dx$

64. Find $\int_0^7 \frac{1}{\sqrt{3x+4}} dx$

65. Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx$

66. Find $\int_0^{\pi} cos^{2023}x\,dx$

67. Find $\int_0^{\pi} \frac{1}{1+\sin x}dx$

68. Find $\int_0^{1} \frac{1}{e^x+e^{-x}}dx$

69. Find $\int_{-2}^{1} \frac{|x|}{x}dx$

70. Find $\int_{\pi}^{2\pi} |\sin x|\,dx$

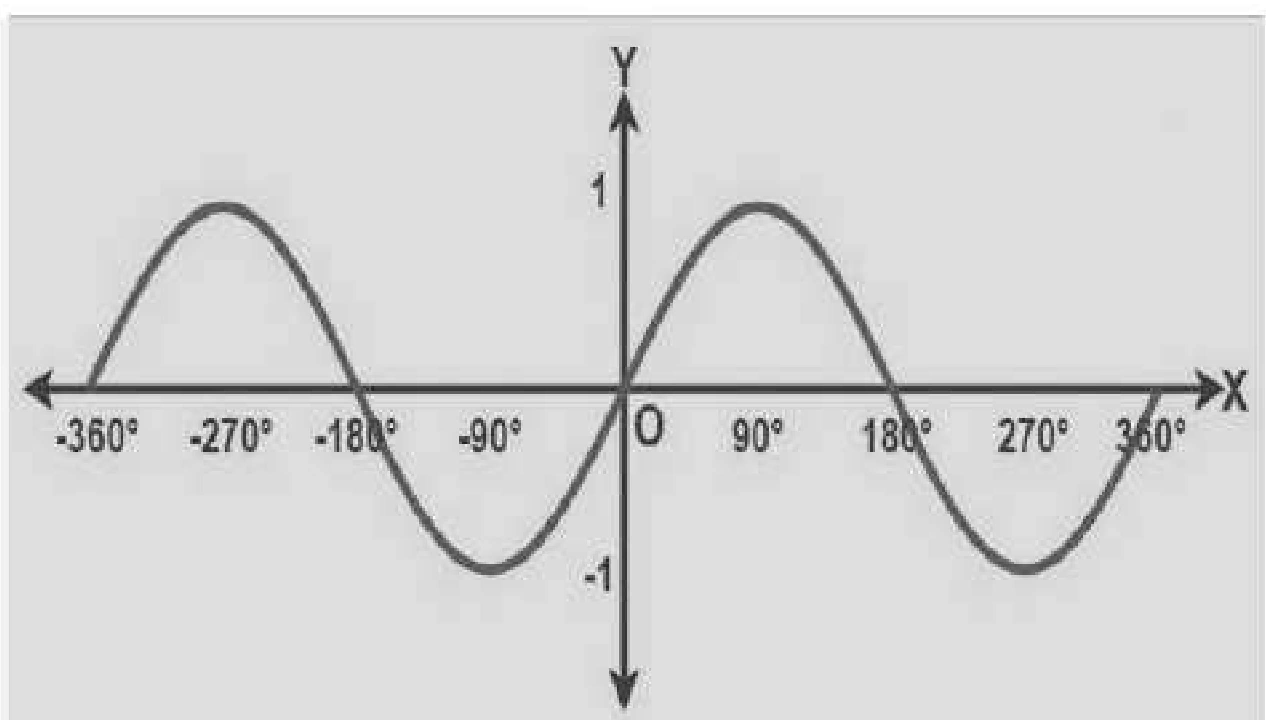
71. Find $\int_0^{2\pi} |\cos x|\,dx$

72. Find $\int_0^{2\pi} |\sin x|\,dx$

73. Find $\int_0^{\pi} \{\cos x + |\cos x|\}dx$

74. Find $\int_0^{1} |2x-1|\,dx$

75. The graph of a trigonometric function f(x) is given below. Find the value of $\int_0^{\frac{\pi}{2}} F(x)\,dx$, where $\frac{d[f(x)]}{dx} = F(x)$



ANSWERS

Q. No.	Answer	Q. No.	Answer
1	$\frac{x^2}{2}+5x+C$	17	$\frac{\sin x}{2}-\frac{\sin 5x}{10}+C$
2	$\frac{(x-1)^3}{3}+C$	18	$x\cos a-\sin a.\log\vert\sin(x+a)\vert+C$
3	$\log\vert\sin x\vert+C$	19	$\frac{\cos 3x}{3}+C$
4	$\frac{(x-\log x)^2}{2}+C$	20	$\frac{\sin 3(2x+1)}{6}+C$
5	$\frac{(\log\sin 2x)^2}{4}+C$	21	$\log\vert\log f(x)\vert+C$
6	$x.\log 3x-x+C$	22	$2\sqrt{f(x)}+C$
7	$\frac{(\log\vert cosec\,x-\cot x\vert)^2}{2}+C$	23	$\log\vert\log\sin x\vert+C$
8	$\frac{e^{sin^2x}}{2}+C$	24	$2e^{tan^{-1}\sqrt{x}}+C$
9	$\frac{x^3}{3}-x+2tan^{-1}x+C$	25	$log_2 e\,.sin^{-1}(2^x)+C$
10	$-\log\vert\sin x+\cos x\vert+C$	26	$\frac{sin^4 x^2}{8}+C$
11	$\frac{2(ax+b)^{\frac{3}{2}}}{3a}+C$	27	$x+C$
12	$2\log\vert 1+\sqrt{x}\vert+C$	28	$\log\vert\frac{1}{\sin x+\cos x}\vert+C$
13	$sin^{-1}x+\sqrt{1-x^2}+C$	29	$\sin x-\cos x+C$
14	$-\cot\frac{x}{2}+C$	30	$\log\vert\frac{2+\cos x}{1+\cos x}\vert+C$
15	$2\sin x-\log\vert\sec x+\tan x\vert+C$	31	$\frac{tan^{-1}(x+1)}{2}+C$
16	$\frac{x}{2}+\frac{\sin 2nx}{4n}+C$	32	$\log\vert\frac{x+1}{x+2}\vert+C$

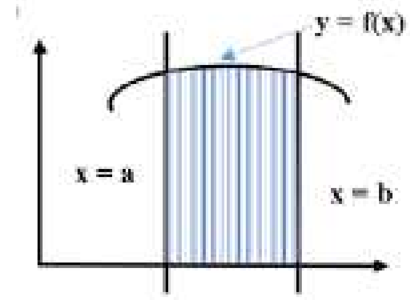
33	$e^x.tan^{-1}x + C$	51	-1
34	$\frac{e^x}{(x-1)^2} + C$	52	$\frac{3}{2}$
35	$e^x \sec x + C$	53	$\frac{\pi}{4}$
36	$\frac{x^2}{2} + C$	54	$\frac{b-a}{2}$
37	$x^2 + C$	55	$\frac{\pi}{4}$
38	$\tan x - \frac{cot^5 x}{5} + C$	56	$\frac{\pi}{8}$
39	$\frac{x^2}{2} + C$	57	0
40	$\sec x + \tan x + C$	58	0
41	$\frac{1}{\cos(a-b)}.\log\left\|\frac{\sin(x+a)}{\cos(x+b)}\right\| + C$	59	2
42	$cos^{-1}(\sec\alpha . \cos x) + C$	60	$\log\sqrt{3}$
43	$\frac{a^x e^x}{1+\log a} + C$	61	$\frac{e-1}{2}$
44	1	62	$\frac{\log ab \log\left(\frac{b}{a}\right)}{2}$
45	$\frac{1}{3}$	63	0
46	$\frac{\pi}{16}$	64	2
47	$\frac{19}{3}$	65	$\frac{\log 2}{2}$
48	$e-1$	66	0
49	1	67	2
50	$\frac{\pi}{4} - \frac{1}{2}$	68	$tan^{-1}e - \frac{\pi}{4}$

69	-1	73	2
70	2	74	$\frac{1}{2}$
71	4	75	1
72	4		

CHAPTER 8

APPLICATION OF INTEGRALS

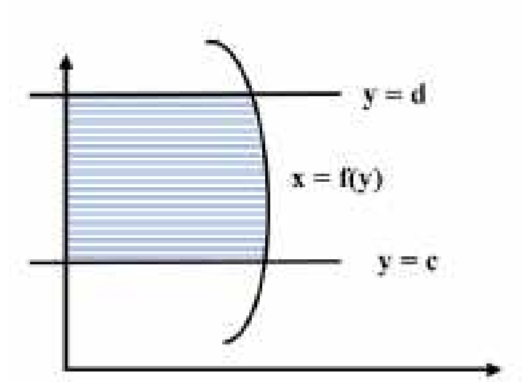
POINTS TO REMEMBER:



Area bounded by the curve $y = f(x)$, x-axis

and the ordinates $x = a$ and $x = b$ is given by

A = $\int_a^b y\,dx = \int_a^b f(x)\,dx$



Area bounded by the curve $x = g(y)$, y-axis

and the ordinates $y = c$ and $y = d$ is given by

A = $\int_c^d x\,dy = \int_c^d g(y)\,dy$

Remarks: If the integral while calculating the area is negative, then we take its absolute value.

Shortcut to find area of some particular curves:

1. Area between $y^2 = 4ax$ and y = mx is $\frac{8a^2}{3m^3}$ sq.units.
2. Area between $y^2 = 4ax$ and y = $mx + c$ is $\frac{72a^2}{m^3}$ sq. units.
3. Area between $x^2 = 4by$ and y = mx is $\frac{8}{3}$ b^2m^3sq.units.
4. Area between $x^2 = 4by$ and y = $mx + c$ is $72\,b^2m^3$sq.units.

5. Area between $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16ab}{3}$ sq. units.

6. Area between $y^2 = 4ax$ and its latus rectum is $\frac{8a^1}{3}$ sq. units.

7. Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq. units.

   Where a, b, m and c are non-zero constants.

QUESTIONS:

1. Find area of the region bounded by $y - x = 0, x = 4, y = 0$.
2. Find area of the region bounded by the parabola $y^2 = 8x$ and its latus rectum.
3. Find area of the region bounded by $y = |x|, y = 1$ *and* y-axis.
4. Find area of the region bounded by the parabola $y^2 = x,\ y = 3$ *and* y-axis.
5. Find area of the region bounded by $y = |x - 1|, x = 1, x = 2$ *and* x-axis.
6. Find area enclosed by the curve $y = cosx$, between $x = 0$ *and* $x = \frac{\pi}{2}$.
7. Find area of the region bounded by the ellipse $\frac{x^1}{36} + \frac{y^1}{25} = 1$.
8. Find area of the region bounded by the curve $x^2 + y^2 = 4$.
9. Find area bounded by the curve $y = cosx, y = sinx$ *and* x-axis between x = 0 and x = $\frac{\pi}{2}$
10. Find area enclosed by the curve $y = sinx$, between $x = 0$ *and* $x = \pi$.
11. Find area of the region bounded $y^2 = 4x$ and $x^2 = 8y$.
12. Find area of the region bounded $y = \sqrt{x}$ and $x = \sqrt{y}$.
13. Find area of the region bounded $y^2 = 4x$ and $y = 3x$.
14. Find area of the region bounded by $x^2 = 8y$ and $y = |x|$.
15. Find area of the region bounded by $y^2 = 2x$ and $x = 2$.
16. Find area of the region bounded by $y = 2x - x^2$ and x-axis.
17. Find area of the region bounded by $y = 4 - x^2$, x-axis and the lines $x = 0$ *and* $x = 2$.
18. Find area of the region bounded by $y = x^2$ and $y = |x|$.

19. Find area of the region bounded by $y^2 = 2y - x$ and y-axis.

20. Find area bounded by the curve $x = 2cost,\ y = sint\ and$ x-axis in the first quadrant.

21. If the area bounded by $y = ax^2$ & $x = ay^2, a > 0$ is 1, then find the value of a.

22. Find area of the region bounded by $y = sinx\ and$ x-axis between x = 0 and x = 2π.

23. Find area of the region bounded by $y^2 = x$, $x = 1$, $x = 4$ and x-axis.

24. Find area of the region bounded by the lines $= 0$, $y = 0$ and $x = y + 2$.

25. Find area of the region bounded by the curve $y = x^3$, $x = 1$, $x = 2$ and x -axis.

26. Find area of the region bounded between $y^2 = 4x$ and $x = 3$.

27. Find area of the region bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ & $x = 2$.

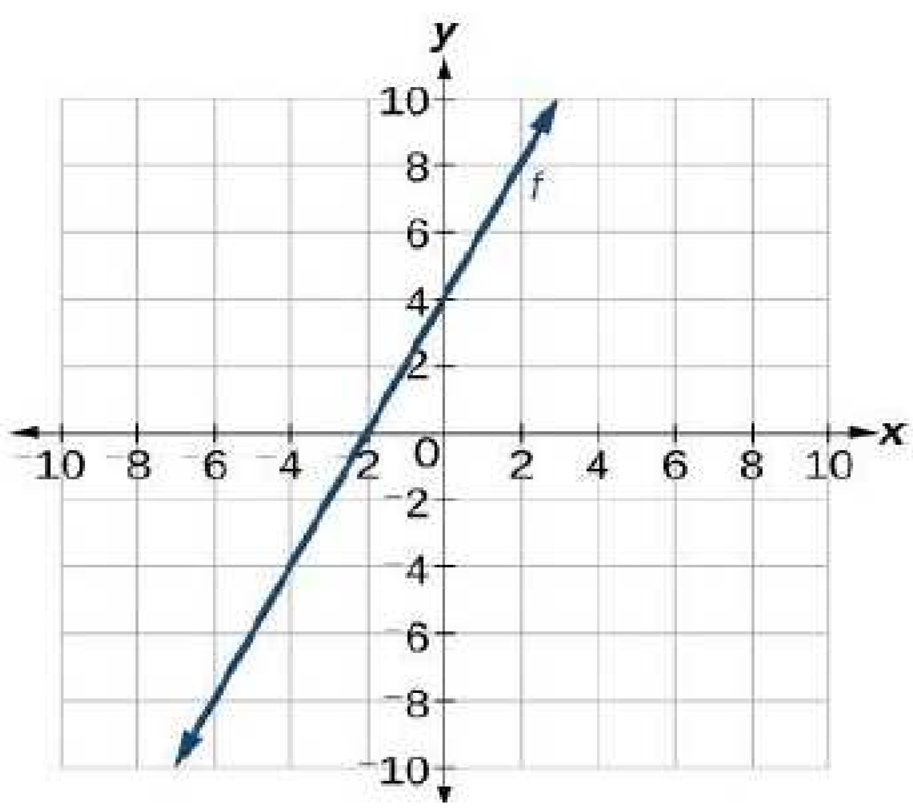
28. Find area lying between $y^2 = 4x$ and $y = 2$x.

29. Find area of the ellipse $4x^2 + 9y^2 = 36$.

30. Find area of the region in first quadrant bounded by the curve $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

31. Find area bounded by the line x + y = 2, x = 0 & y = 0.

32. The graph of a function f(x) is given below. Find the value of $\int_{-2}^{0} f(x)\,dx$.



33. Find area of the region bounded by the lines $|x| + |y| = 1$.

34. Find area of the region bounded by the curve $y = x\,|x|$, x-axis and the lines $x = -1$ and $x = 1$

35. Find area of the region bounded by $y = |x - 2|$, $x = 1$, $x = 3$ *and* x-axis.

ANSWERS

Q. No.	Answer	Q. No.	Answer
1	8 square units	19	$\frac{4}{3}$ square units
2	$\frac{32}{3}$ square units	20	$\frac{\pi}{2}$ square units
3	$\frac{1}{2}$ square units	21	$\frac{\sqrt{3}}{3}$
4	9 square units	22	4 square units
5	$\frac{1}{2}$ square units	23	$\frac{14}{3}$ square units
6	1 square unit	24	2 square units
7	30π square units	25	$\frac{15}{4}$ square units
8	4π square units	26	$8\sqrt{3}$ square units
9	$(2-\sqrt{2})$ square units	27	2π square units
10	2 square units	28	$\frac{1}{3}$ square units
11	$\frac{32}{3}$ square units	29	6π square units
12	$\frac{1}{3}$ square units	30	$\frac{7}{3}$ square units
13	$\frac{8}{81}$ square units	31	2 square units
14	$\frac{64}{3}$ square units	32	4 square units
15	$\frac{16}{3}$ square unit	33	2 square units
16	$\frac{4}{3}$ square units	34	$\frac{2}{3}$ square units
17	$\frac{16}{3}$ square units	35	1 square unit
18	$\frac{1}{3}$ square units		

CHAPTER 9

DIFFERENTIAL EQUATIONS

POINTS TO REMEMBER:

1. **Differential equation:**

   An equation involving derivative (derivatives) of the dependent variable with respect to independent variable (variables) is called a differential equation.

   A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation, e.g.,

   $\frac{dy}{dx} + 6y = 2e^x$ **and** $\frac{d^1y}{dx^1} - \left(\frac{dy}{dx}\right)^2 + y = \cos x$

2. **Order and degree of differential equation:**

   Order of a differential equation is defined as the order of the highest ordered derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

   The integral power of the highest ordered derivative involved in a differential equation when expressed as a polynomial in dependent variable and its derivatives is called the degree of the differential equation.

3. **Forming a differential equation:**

   Method- Suppose an equation $f(x, y, c_1, c_2, c_3 \dots c_n) = 0$ contains n arbitrary constants or parameters is being given. It represents a family of curves and its differential equation will be formed as given below:

   Step1: Differentiate the given equation n times to get n more equations.

   Step 2: Using all these equations, eliminate the constants to get the required D.E.

   Note: If an equation contains n arbitrary constants then order of its D.E. is n but you cannot say anything about its degree.

4. Solution of a Differential equation:

(i) A function of the type f(x,y) satisfying the D.E. is called its solution.

(ii) General solution: If the solution of a D.E. of order n contains n arbitrary constants, then the solution is called the general solution or complete integral.

(iii) Particular solution: A solution obtained by giving particular values to arbitrary constants in the general solution, is called a particular solution or particular integral.

5. Types of Differential equation:

(i) Differential equations in which variables can be separable.

A first order-first degree differential equation is of the form

$\frac{dy}{dx} = F(x, y)$

If F (x, y) can be expressed as a product f (x) g(y), where, $f(x)$ is a function of x and $g(y)$ is a function of y, then the differential equation is said to be of variable separable type.

The differential equation $\frac{dy}{dx} = F(x, y)$, then has the form $\frac{dy}{dx} = f(x)g(y)$

(ii) Homogeneous differential equations

A function F(x, y) is said to be homogeneous function of degree n if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non-zero constant λ.

or

A function F (x, y) is a homogeneous function of degree n if

F $(x,y) = x^n$ g $(\frac{y}{x})$ or y^n g $(\frac{x}{y})$.

(iii) Linear differential equations

A differential equation of the form $\frac{dy}{dx} + Py = Q$ where, P and Q are constants or functions of x only, is known as a first order linear differential equation.

The function g(x) = $e^{\int P dx}$ is called Integrating Factor (I.F.) of the given differential equation.

QUESTIONS:

Find the order and degree of each of the following differential equations:

1. $(\frac{d^2y}{dx^2})^3 + (\frac{dy}{dx})^4 + 6 = (\frac{d^3y}{dx^3})^2$
2. $\left(\frac{dy}{dx}\right)^2 + 5 = \frac{d^2y}{dx^2}$
3. $(\frac{dy}{dx})^2 = (2 + \frac{d^2y}{dx^2})^{\frac{3}{2}}$
4. $y = x\frac{dy}{dx} + \sqrt{(a)^2\left(\frac{dy}{dx}\right)^2 + (b)^2}$
5. $(\frac{d^4y}{dx^4})^2 + cos\left(\frac{d^3y}{dx^3}\right) = 0$
6. $\frac{d}{dx}[(\frac{dy}{dx})^4] + 5 = 0$
7. $y''' + y'' + (e)^{y'} = 0$
8. **How many arbitrary constants are there in the general solution of a differential equation of degree 2 and order 3?**
9. **Find the differential equations whose general solution is** $tan^{-1}x + tan^{-1}y = c.$

   Find the general solution of the differential equations:

10. $\frac{dy}{dx} + \frac{tany}{tanx} = 0$
11. $\frac{dy}{dx} + y\,logy\,cotx = 0$
12. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$
13. $x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$
14. $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$
15. $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$
16. $x^2\frac{dy}{dx} = x^2 + xy + y^2$
17. $x\frac{dy}{dx} = y\,(logy - logx + 1)$
18. $x^2dy + y(x+y)dx = 0$
19. $\frac{dy}{dx} - y = cosx$

20. $\frac{dy}{dx} + y\,tanx = secx$

21. Find the integrating factor of $(1 + x^2)dy + 2xy\,dx = cotx\,dx \ (x \neq 0)$

22. Find the integrating factor of $(1 + y^2)dx = (tan^{-1}y - x)dy$

23. Find the integrating factor of $xlogx\frac{dy}{dx} + y = 2logx$

24. What is the product of order & degree of the differential equation

$(\frac{d^1y}{dx^1})^3 = k(1 + (\frac{dy}{dx})^2)^2$?

25. Find the sum of the order and degree of the differential equation

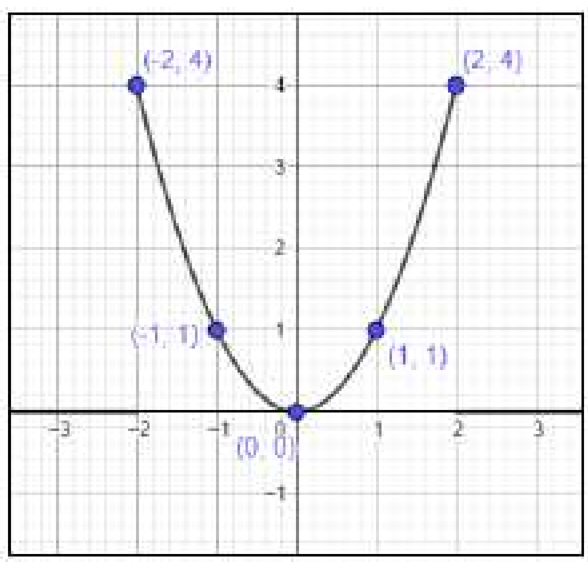
$\frac{d}{dx}[(\frac{dy}{dx})^4] + x = 0$

26. What is the integrating factor of $(x + 3y^2)\frac{dy}{dx} = y, \ y > 0$?

27. Find the general solution of the differential equation $\frac{ydx - xdy}{y} = 0$.

28. If the graph of a function y = f(x) is given below such that $\left(\frac{dy}{dx}\right) = 2x$.

Find the value of f(1.1).



29. If y = f(x) is the solution of differential equation $\left(\frac{dy}{dx}\right) = 2, f(0) = 0$, then find the value of $f(5)$.

30. Find the general solution of the differential equation $\frac{dy}{dx} = xy + x + y + 1$

ANSWERS

Q. No.	Answer	Q. No.	Answer
1	Degree 2 and order 3	16	$tan^{-1}\left(\frac{y}{x}\right) = log\|x\| + c$
2	Degree 1 and order 2	17	$log\left\|\frac{x}{y}\right\| = cx$
3	Degree not defined and order 2	18	$y + 2x = c^2x^2y$
4	Degree not defined and order 1	19	$y = \frac{sinx - cosx}{2} + ce^x$
5	Degree not defined and order 4	20	$y = sinx + c\,cosx$
6	Degree 1 and order 2	21	$1 + x^2$
7	Degree not defined and order 3	22	$e^{tan^{-1}y}$
8	3	23	$logx$
9	$(1 + x^1)\frac{dy}{dx} + (1 + y^1) = 0$	24	6
10	$sinx\,siny = c$	25	3
11	$sinx\,log\|y\| = c$	26	$\frac{1}{y}$
12	$sin^{-1}x + sin^{-1}y = c$	27	$y = cx$
13	$\sqrt{(1 + x^2)} + \sqrt{(1 + y^2)} = c$	28	1.21
14	$log\left\|\frac{x}{y}\right\| = \frac{1}{x} + \frac{1}{y} + c$	29	10
15	$(x - y) + log\|x + y\| = c$	30	$log\|y + 1\| = \frac{1}{2}x^2 + x + c$

CHAPTER 10

VECTOR ALGEBRA

POINTS TO REMEMBER:

1. Scalars: Real numbers are called scalar.
2. Vectors:

(i) A directed line segment $\overrightarrow{AB}$ is called a vector.

(ii) A vector $\overrightarrow{AB}$ has magnitude and direction.

(iii) Its length or magnitude is denoted by $|\overrightarrow{AB}|$ and its direction is from A to B.

(iv) If $\vec{\alpha} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$, then in short we denote it by (a, b, c). Thus we say that the components of $\vec{\alpha}$ along x-axis, y-axis and z-axis are a, b and c respectively.

(v) $\vec{\alpha} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$, then magnitude of $\vec{\alpha} = |\vec{\alpha}| = \sqrt{a^2 + b^2 + c^2}$

(vi) Collinear vectors: Vectors along the same straight line are called collinear vectors.

- $\vec{a}$ and $\vec{b}$ are collinear $\Leftrightarrow \vec{a} = \alpha\,\vec{b}$ for some α.
- The necessary condition for $\vec{a}$ and $\vec{b}$ to be collinear is $\vec{a} \times \vec{b} = \vec{0}$

(vii) Position vector of a point: Let O be the origin and P(x, y, z) be a point then

$\overrightarrow{OP} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ is called the position vector of P.

(x) The unit vector of $\overrightarrow{OP}$ denoted by $\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$

(xi) Consider the position vector $\overrightarrow{OP}$ $(\vec{r})$ of a point P(x, y, z). The angles α, β, γ made by the vector with the positive directions of x, y and z-axes respectively, are called its direction angles. The cosine values of these angles, i.e., cosα, cosβ and cos γ are called direction cosines of the vector, and usually denoted by l, m and n respectively.

The numbers lr, mr and nr, proportional to the direction cosines are called as direction ratios of vector, and denoted as a, b and c respectively.

If $\overrightarrow{OP} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, then direction ratios of $\overrightarrow{OP}$ are x, y and z and direction cosines of the vector $\overrightarrow{OP}$ are $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$

3. **Section formula:**

(i) Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point which divides AB is in the ratio m:n (internally) , then position vector of P = $\frac{m\vec{b} + n\,\vec{a}}{m+n}$

(ii) Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point which divides AB is in the ratio m:n (externally) , then position vector of P = $\frac{m\vec{b} - n\,\vec{a}}{m-n}$

(iii) Position vector of mid-point of AB= $\frac{1}{2}\left(\vec{a} + \vec{b}\right)$

(iv) Let the position vector of vertices of a triangle ABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, then position vector of centroid = $\frac{1}{3}\left(\vec{a} + \vec{b} + \vec{c}\right)$

(v) Points A, B and C will be collinear $\Leftrightarrow \overrightarrow{AB} = t\overrightarrow{AC}$ for some scalar t.

(vi) Points A, B, C and D be coplanar $\Leftrightarrow \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.

(vii) If $\vec{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$, $\vec{b} = b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\,\hat{k}$ and $\vec{c} = c_1\,\hat{\imath} + c_2\,\hat{\jmath} + c_3\,\hat{k}$ will be coplanar $\Leftrightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

4. **Scalar multiple of a vector:** Let $\vec{a}$ be a given vector and λ is a scalar. Then the product of the vector $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$ is called multiplication of vector by the scalar λ .

5. **Scalar or dot product of vectors:**

The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\,.\,\vec{b}$, is defined as

$\vec{a}\,.\,\vec{b} = |\vec{a}|\,.\,|\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $0 \le \theta \le \pi$.

Observations:

(i) $\cos\theta = \dfrac{\vec{a}\,.\vec{b}}{|\vec{a}|\,.|\vec{b}|}$

(ii) $\vec{a}$ and $\vec{b}$ are perpendicular, if and only if $\vec{a}\,.\,\vec{b} = \mathbf{0}$.

(iii) $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$

(iv) $\vec{a}\,.\,\vec{a} = |\vec{a}|^2$

(v) θ is acute $\Leftrightarrow \vec{a}\,.\,\vec{b} > 0$

(vi) Length of projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\,.\vec{b}}{|\vec{b}|}$

(vii) If $\vec{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\,\hat{k}$

then $\vec{a}\,.\,\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

6. **Vector or cross product of vectors:**

The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}|\,.\,|\vec{b}|\sin\theta\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and $0 \le \theta \le \pi$ where $\hat{n}$ is the unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

Observations:

(i) $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,.|\vec{b}|}$

(ii) Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel, if and only if $\vec{a} \times \vec{b} = \mathbf{0}$.

(iii) $\vec{a} \times \vec{b} = -\,(\vec{b} \times \vec{a})$

(iv) If $\vec{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\,\hat{k}$

then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(v) If $\vec{a}$ and $\vec{b}$ are two adjacent sides of a triangle then its area= $\frac{1}{2}$ $|\vec{a} \times \vec{b}|$ square units i.e. Area of triangle ABC is $\frac{1}{2}$ $|\overrightarrow{AB} \times \overrightarrow{AC}|$ square units.

(vi) If $\vec{a}$ and $\vec{b}$ are two adjacent sides of a parallelogram then its area is $|\vec{a} \times \vec{b}|$ square units.

(vii) If $\vec{c}$ and $\vec{d}$ are the diagonals of a parallelogram then its area = $\frac{1}{2}$ $|\vec{c} \times \vec{d}|$ square units.

(viii) If $\vec{a}$ and $\vec{b}$ are two adjacent sides of a parallelogram then the length of its diagonals are $|\vec{a} + \vec{b}|$ units and $|\vec{a} - \vec{b}|$units.

QUESTIONS:

1. Find the magnitude of the vector $6\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.
2. Find a vector whose initial and terminal points are (2, 5, 0) and (-3, 7, 4) respectively.
3. Find the projection of the vector $\vec{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ along the vector $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.
4. Find the projection of the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ along vector $\hat{\jmath}$.
5. If $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\vec{b} = 5\hat{\imath} - 3\hat{\jmath} + \hat{k}$, then find the projection of $\vec{b}$ on $\vec{a}$.
6. If the projection of $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ on $\vec{b} = 2\hat{\imath} + \mu\hat{k}$ is zero, then find the value of μ.
7. If $\vec{a} \cdot \vec{b} = \frac{1}{2}|\vec{b}|\,|\vec{a}|$, then find the angle between $\vec{a}$ and $\vec{b}$.
8. Find the angle between vectors $\hat{\imath} - \hat{\jmath}$ and $\hat{\jmath} - \hat{k}$.
9. Find a vector in the direction of the vector $\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and magnitude 3 units.
10. If $\vec{a}$ = (1,-1) and $\vec{b} = (-2, m)$ are collinear vectors, then find the value of m.
11. If $|\vec{a}| = 4$ and $-3 \le k \le 2$, then find the range of $|k\vec{a}|$.

12. Find the value of λ , for which the vectors $3\,\hat{\imath} - 6\,\hat{\jmath} + \hat{k}$ and $2\,\hat{\imath} - 4\,\hat{\jmath} + \lambda\hat{k}$ are parallel.

13. If the sides AB and AD of a parallelogram ABCD are represented by the vectors $2\,\hat{\imath} + 4\,\hat{\jmath} - 5\,\hat{k}$ and $\hat{\imath} + 2\,\hat{\jmath} + 3\hat{k}$, then find a unit vector along $\overrightarrow{AC}$.

14. If three points A, B and C with position vectors $\hat{\imath} + x\,\hat{\jmath} + 3\,\hat{k}$, $3\,\hat{\imath} + 4\,\hat{\jmath} + 7\,\hat{k}$ and $y\,\hat{\imath} - 2\,\hat{\jmath} - 5\hat{k}$ respectively are collinear, then find the value of $(x\,,y)$.

15. Find the value of λ for which the vectors $2\hat{\imath} + \lambda\,\hat{\jmath} + \hat{k}$ *and* $\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$ are perpendicular to each other.

16. Find the position vector of the point which divides the line segment joining the points with position vectors $2\vec{a} - 3\,\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio 3:1 (internally).

17. If A, B, C, D and E are five coplanar points , then find the value of $\overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC} + \overrightarrow{AE} + \overrightarrow{BE} + \overrightarrow{CE}$.

18. Find the value of $(\vec{a}.\hat{\imath})^2 + (\vec{a}.\hat{\jmath})^2 + (\vec{a}.\hat{k})^2$.

19. If $|\vec{a}| = 3$ and $|\vec{b}| = 4$, then find the value of λ for which $\vec{a} + \lambda\,\vec{b}$ and $\vec{a} - \lambda\,\vec{b}$ are perpendicular.

20. If $(\vec{a}.\hat{\imath}) = \vec{a}\,.(\hat{\imath} + \hat{\jmath}) = \vec{a}.(\,\hat{\imath} + \hat{\jmath} + \hat{k}\,) = 1$, then find the value of $\vec{a}$.

21. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$ then find the range of θ when $\vec{a}\,.\,\vec{b} \geq 0$.

22. If the points A, B and C with position vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\,\hat{\jmath} - 5\,\hat{k}$ and $\alpha\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$ respectively are the vertices of a right angle triangle with $\angle C = \pi/2$, then find the value of α.

23. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 4 respectively and $\vec{a}\,.\,\vec{b} = 2\sqrt{3}$.

24. If $\vec{a}$, $\vec{b}$ and $\sqrt{3}\,\vec{a} - \vec{b}$ are unit vectors , then find the angle between $\vec{a}$ and $\vec{b}$.

25. If OACB is a parallelogram with $\overrightarrow{OC} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$, then find the value of $\overrightarrow{OA}$.

26. If the angle between the vectors $\hat{\imath} + \hat{k}$ and $\hat{\imath} + \hat{\jmath} + \lambda\hat{k}$ is $\pi/3$, then find the value of λ.

27. If ABCD is a rhombus whose diagonals intersect at E, then find the value of $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED}$.

28. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$ then find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

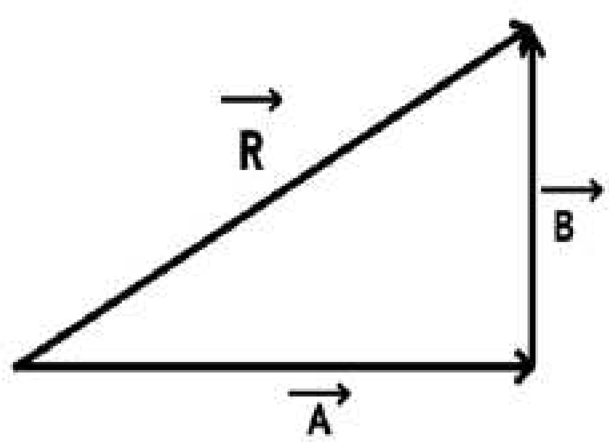
29. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors satisfying the condition $\vec{a} + \vec{b} + \vec{c} = 0$. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 2$, then find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

30. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} - 7\hat{\jmath} + \hat{k}$.

31. If $\overrightarrow{AB} \times \overrightarrow{AC} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$, then find the area of triangle ABC.

32. Find the area of a triangle formed by the vertices O, A and B where $\overrightarrow{OA} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\overrightarrow{OB} = -3\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

33. In the given figure, If $\vec{A}, \vec{B}$ & $\vec{R}$ represents the sides of a triangle such that $\vec{A} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{B} = \hat{\imath} + \hat{k}$, then find $|\vec{R}|$.



34. If $\overrightarrow{OA} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\overrightarrow{OB} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$, then find the vector which bisect $\angle$AOB.

35. Find the value of $\vec{r}$ whose magnitude is $\sqrt{2}$ units and which makes angle of $\pi/4$ and $\pi/2$ with y-axis and z-axis respectively.

36. Find the unit vector perpendicular to the vectors $i - \hat{\jmath}$ and $\hat{\imath} + \hat{\jmath}$.

37. Find area (in square units) of the triangle having vertices with position vectors $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, $-2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $4\hat{\imath} - 7\hat{\jmath} + 7\hat{k}$ respectively.

38. $\vec{a}$, $\vec{b}$ are two unit vectors and θ is the angle between them. If $\vec{a}+\vec{b}$ is a unit vector, then find the value of θ.

39. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$, then find the angle between $\vec{a}$ and $\vec{b}$.

40. Find the number of unit vectors perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

41. If $\vec{a} \cdot \vec{a} = |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 144$ and $|\vec{a}| = 4$, then find the value of $|\vec{b}|$.

42. If $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 12$, then find the value of $|\vec{a} \times \vec{b}|$.

43. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$, then find the value of $\vec{b}$.

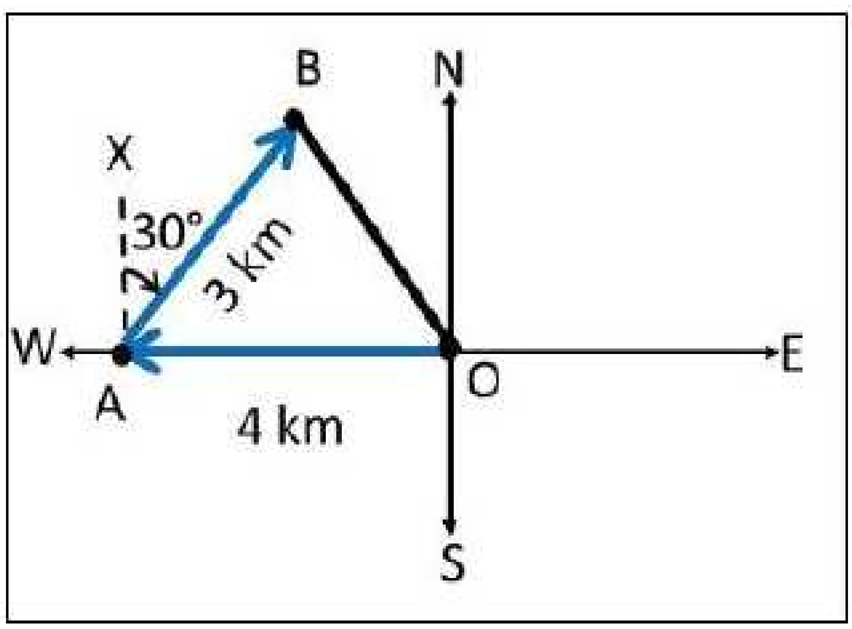
44. If the position vector of a point A is $\vec{a} + 2\vec{b}$ and P with position vector $\vec{a}$ divides a line segment AB in the ratio 2:3, then find the position vector of the point B.

45. If $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a} \times \vec{b}| = 1$, then find the angle between $\vec{a}$ and $\vec{b}$.

46. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{a} \times \vec{b}| = 10$, then find the value of $|\vec{a} \cdot \vec{b}|^2$.

47. If $|\vec{a} \times \vec{b}| = 4$ and $|\vec{a} \cdot \vec{b}| = 2$, then find the value of $|\vec{a}|^2|\vec{b}|^2$.

48. If O represents the Origin, WE represents the x-axis & NS represents y-axis, then Find $\vec{AB}$.



49. If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle α, then find the value of $|\vec{a} - \vec{b}|$.

50. If $\vec{a} = \hat{i} + \hat{j} + p\hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$, then find the value of p.

ANSWERS

Q. No.	Answer	Q. No.	Answer
1	7 units	21	$[0, \pi/2]$
2	$-5\hat{i} + 2\hat{j} + 4\hat{k}$	22	2,1
3	$\frac{2}{3}$	23	$\pi/3$
4	1	24	$\pi/6$
5	3	25	$\frac{1}{2}(\vec{a} - \vec{b})$
6	$-\frac{2}{3}$	26	$0, -4$
7	60^0	27	$\vec{0}$
8	$\frac{2\pi}{3}$	28	$-\frac{3}{2}$
9	$\hat{i} - 2\hat{j} + 2\hat{k}$	29	$-\frac{29}{2}$
10	2	30	$15\sqrt{2}$ square units
11	[0,12]	31	3 square units
12	$\frac{2}{3}$	32	$3\sqrt{5}$ square units
13	$\frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$	33	3 *units*
14	(2,-3)	34	$4\hat{i} + 2\hat{j}$
15	$-\frac{5}{2}$	35	$\pm(\hat{i} + \hat{j})$
16	$\frac{5\vec{a}}{4}$	36	$\pm\hat{k}$
17	$3\overrightarrow{DE}$	37	0
18	$\lvert\vec{a}\rvert^2$	38	$2\pi/3$
19	$\pm\frac{3}{4}$	39	90^0
20	$\hat{i}$	40	2

41	3	46	44
42	16	47	20
43	$\hat{\imath}$	48	$\frac{3}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$
44	$\vec{a} - 3\vec{b}$	49	$2 \sin\alpha/2$
45	$\pi/2$	50	1

Chapter 11

THREE DIMENSIONAL GEOMETRY

POINTS TO REMEMBER

Direction cosines: If a directed line L passing through the origin make angles α, β and γ with x, y and z-axes, respectively called direction angles, then cosine of these angles namely cos α, cos β and cos γ are called direction cosines of the directed line L.

Direction ratios: Any three numbers which are proportional to the direction cosines of a line are called the direction ratios of the line. If l, m and n are direction cosines and a, b, c are direction ratios of a line, then a = λl, b=λm and c = λn, for any nonzero λ ∈ R.

The direction ratios of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as: $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ or $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$

EQUATION OF A LINE THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

Vector equation of a line:

Let $\vec{a}$ be the position vector of the given point A with respect to the origin O of the rectangular coordinate system. Let l be the line which passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the line, then the vector equation of the line is given by $\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is some real number.

Cartesian equation of a line:

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b, c, then cartesian equation of the line is given by

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

If l, m and n are the direction cosines of the line, then the equation of the line is

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

Vector equation of a line:

Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) respectively that are lying on a line. Let $\vec{r}$ be the position vector of an arbitrary point $P(x, y, z)$ on the line then the vector equation of the line is given by

$\vec{r} = \vec{a} + \lambda\ (\vec{b} - \vec{a})$ where λ is some real number.

Cartesian equation of the line: $\frac{x-x_1}{x_1-x_1} = \frac{y-y_1}{y_1-y_1} = \frac{z-z_1}{z_1-z_1}$

ANGLE BETWEEN TWO LINES

Let l_1 and l_2 be two lines passing through the origin and with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively. The angle θ between them is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively are

(i) perpendicular i.e. if θ = 90° when $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) parallel i.e. if θ = 0 , when $\frac{a_1}{a_1} = \frac{b_1}{b_1} = \frac{c_1}{c_1}$

DISTANCE BETWEEN TWO SKEW LINES

Let l_1 and l_2 be two skew lines with equations $\vec{r} = \vec{a_1} + \lambda\ \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\ \vec{b_2}$

The required shortest distance is $\left|\frac{(\vec{b_1} \times \vec{b_1}) \cdot (\vec{a_1} - \vec{a_1})}{|\vec{b_1} \times \vec{b_1}|}\right|$ units.

DISTANCE BETWEEN PARALLEL LINES

Let l_1 and l_2 be two skew lines with equations $\vec{r} = \overrightarrow{a_1} + \lambda\ \vec{b}$ and $\vec{r} = \overrightarrow{a_2} + \mu\ \vec{b}$

The required shortest distance is $\left| \frac{\vec{b} \times (\overrightarrow{a_1} - \overrightarrow{a_1})}{|\ \vec{b}\ |} \right|$ units.

QUESTIONS:

1. If a line makes angle of 90^0, 135^0 and 45^0 with x, y and z axes respectively, then find its direction cosines.
2. If a line has direction ratios 2, -3 and - 6 , then determine its direction cosines.
3. Find the direction cosines of x, y and z axes.
4. Find the direction cosines of the line passing through the two points (3, 5,-4) and (-1, 1, 2).
5. Find the vector equation of the line passing through the point (5, 2,-4) and parallel to the vector $3\ \hat{\imath} + 2\ \hat{\jmath} - 8\hat{k}$.
6. Find the cartesian equation of the line passing through the point (1, 2, 3) and parallel to the vector $3\ \hat{\imath} + 2\ \hat{\jmath} - 2\hat{k}$.
7. Find the cartesian equation of the line passing through the point with position vector $2\ \hat{\imath} - \hat{\jmath} + 4\hat{k}$ and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$
8. The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Find its vector form.
9. The vector equation of a line is $\vec{r} = (\hat{\imath} + 2\ \hat{\jmath} - 4\hat{k}) + \lambda(2\hat{\imath} + 3\ \hat{\jmath} - \hat{k})$. Find its cartesian form.
10. Find the vector equation of the line that passes through the origin and (5, -2, 3).
11. Find the Direction ratio's of the line that passes through the origin and (-1, 2, 4).
12. Find the vector equation of the line that passes through the points (3, -2, -5) and (3, -2, 6).

13. Find the cartesian equation of the line that passes through the points (2, -1, 3) and (2, -4, 3).

14. Find the angle between the lines 2x = 3y = -z and 6x = -y = -4z.

15. If a line makes angles $\pi/3$ and $\pi/4$ with the x-axis and y-axis respectively, then find the acute angle made by the line with z-axis.

16. If a line is equally inclined with the co-ordinate axes, then find its direction cosines.

17. Find the equation of x-axis.

18. Find the coordinates of foot of the perpendicular drawn from the point P(2,-3,4) on y-axis.

19. Find the distance of the point P(a, b, c) from x-axis.

20. Find length of the perpendicular drawn from the point (4, -7, 3) on y-axis.

21. If the direction cosines of a line are <k, k, k> then find the value of k.

22. If the direction angles of a line are a, b and c respectively, then find the value of $\sin^2 a + \sin^2 b + \sin^2 c$.

23. If the direction angles of a line are α, β and γ respectively, then find the value of $\text{Cos}2\alpha + \cos 2\beta + \cos 2\gamma$.

24. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{6-z}{5}$ are mutually perpendicular, then find the value of k.

25. Find the value of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

26. The cartesian equation of a line are 6x-2 = 3y+1 = 2z-2. Find its direction ratios.

27. Find the direction ratios of the line perpendicular to the lines $\frac{x-3}{2} = \frac{y+7}{-3} = \frac{z-2}{1}$ and $\frac{x+2}{1} = \frac{y+3}{2} = \frac{z-5}{-2}$.

28. Find the equation of the line in vector form passing through the point (-1, 3, 5) and parallel to the line $\frac{x-3}{2} = \frac{y-4}{3}$, z = 2.

29. O is the origin and P is a point at a distance of 7 units from the origin. If the direction ratios of OP are < 3, -2, 6 >, then find the coordinates of P.

30. Find the length of the perpendicular drawn from the point P (1, -1, 2) on the line $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z+2}{4}$.

31. A point P lies on the line segment joining the points (-1, 3, 2) and (5, 0, 6). If the x-coordinate of P is 2, then find its z- coordinate.

32. Find the direction ratios and direction cosines of a line parallel to the given line $6x - 12 = 3y + 9 = 2z - 2$.

33. Find the angle between the lines $\vec{r} = (4\hat{\imath} - \hat{\jmath}) + \lambda(2\hat{\imath} + \hat{\jmath} - 3\hat{k})$ and $\vec{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \mu(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})$.

34. Find the angle between the lines $\frac{x+4}{1} = \frac{y-3}{2} = \frac{z+2}{3}$ and $\frac{x}{3} = \frac{y}{-2} = \frac{z}{1}$.

35. Find the ratio in which the line segment joining the points (-2, 4, 5) and (3, 5, -4) is divided by the line x = 0.

36. Find the angle between a line with the direction ratios < 2, 2, 1 > and a line joining the points (3, 1, 4) and (7, 2, 12).

37. Find the coordinates of foot of perpendicular drawn from the point (-2, 8, 7) on XZ plane.

38. If the points (3, 2, 2), (2, 3, 4) and (1, p-2 , 6) are collinear , then find the value of p.

39. If a line joining the points (2,-1, 1) and (3, 1, 5) is perpendicular to the line joining the points (3, α, 7) and (1, 0, 4) , then find the value of α.

40. Find the direction cosines of the line $\frac{4-x}{2} = \frac{3y-6}{3} = \frac{4-2z}{4}$.

41. The equation of the line is $\frac{2x-5}{4} = \frac{y+4}{3} = \frac{6-z}{6}$. Find the direction cosines of a line parallel to this line.

42. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-3}{k} = \frac{y-4}{1} = \frac{z-5}{1}$ are intersecting, then find the value of k.

43. O is the origin and P is a point at a distance of 3 units from the origin. If the direction ratios of OP are < 1, -2, -2>, then find the coordinates of P.

44. Find the angle between the pair of lines $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$.

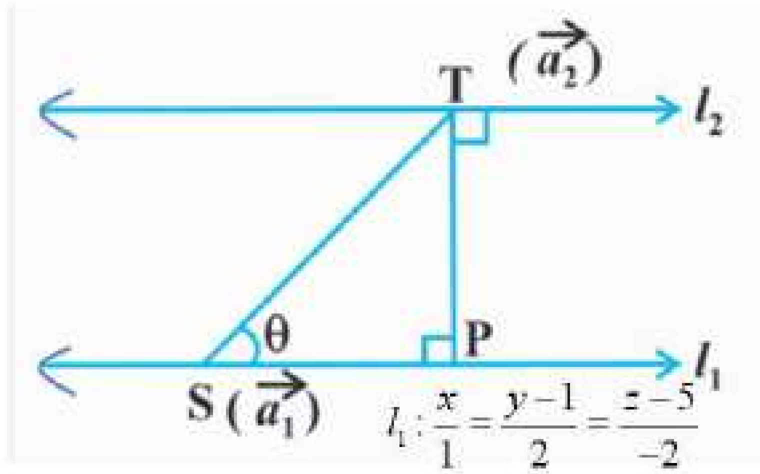
45. Find the angle between the pair of lines given by $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ and $\vec{r} = (5\hat{\imath} - 2\hat{\jmath}) + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$.

46. If P is a point in a space such that $\overrightarrow{OP}$ is inclined to x-axis at 45^0 and y-axis at 60^0, then find its inclination with z-axis.

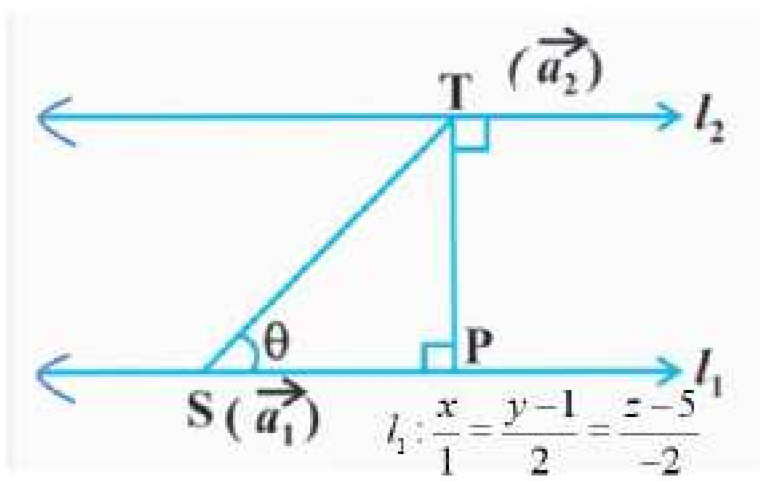
47. A(-1, p, 2), B(-2, 0, -1), C(-5, 0, p) and D(-2, 3p, 3) are four points in space. If lines AB and CD are parallel, then find the value of p.

48. Find the length of perpendicular from the (1, 2, 3) to the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$.

49. If $l_1 \| l_2$ in figure given below, then find the direction cosines of line l_2.



50. If Equation of TP in figure given below is $\frac{x-a}{2}=\frac{y-b}{1}=\frac{z-c}{k}$, then what is the value of k.



ANSWERS

Q. No.	Answer	Q. No.	Answer
1	$0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$	21	$\pm\frac{1}{\sqrt{3}}$
2	$\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}$	22	2
3	(1 , 0 , 0), (0 , 1 , 0) and (0 , 0 , 1)	23	-1
4	$-\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$	24	$-\frac{10}{7}$
5	$\vec{r} = (5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$	25	$\frac{70}{11}$
6	$\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}$	26	1, 2, 3
7	$\frac{x-2}{3} = \frac{y+1}{5} = \frac{z-4}{6}$	27	< 4, 5, 7>
8	$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$	28	$\vec{r} = (-\hat{i} + 3\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j})$
9	$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{-1}$	29	(3 , -2 , 6)
10	$\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$	30	0 unit
11	<-1, 2, 4> Or <-a, 2a, 4a> Where a is a non-zero constant	31	4
12	$\vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + \lambda(11\hat{k})$	32	$(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ or (1, 2, 3) and $(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$
13	$\frac{x-2}{0} = \frac{y+1}{-3} = \frac{z-3}{0}$	33	$\pi/3$
14	90^0	34	$\cos^{-1}(\frac{1}{7})$
15	$\pi/3$	35	2:3
16	$\pm < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$	36	$\cos^{-1}(\frac{2}{3})$
17	$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$	37	(-2,0,7)
18	(0, -3 , 0)	38	6
19	$\sqrt{b^2 + c^2}$ units	39	-7
20	5 units	40	$< -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} >$

41	$<\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}>$	46	60^0 or 120^0
42	1 or -1	47	-6
43	(1, -2, -2)	48	$\sqrt{2}$ units
44	$\cos^{-1}(\frac{8\sqrt{3}}{15})$	49	$<\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}>$ OR $<\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}>$
45	$\cos^{-1}(\frac{19}{21})$	50	2

CHAPTER 12

LINEAR PROGRAMMING

POINTS TO REMEMBER:

The term linear implies that all the mathematical relations used in the problem are linear relations while the term programming refers to the method of determining a particular programme or plan of action.

Linear Programming Problem is one that is concerned with finding the optimal value (maximum or minimum value) of a linear function (called objective function) of several variables (say x and y), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).

Objective functions:- Linear function $Z = ax + by$ where a and b are constraints which has to be maximized or minimized is called a linear objective function.

Constraints:- The linear inequalities or in equations or restrictions on the variables of a linear programming problem.

Optimization problem:- A problem which seeks to maximise or minimise a linear function (say of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an optimization problem.

Feasible region:- It is defined as a set of points which satisfies all the constraints.

Feasible solutions:- Points within and on the boundary of the feasible region represents feasible solutions of the constraints.

Optimal feasible solution:- Feasible solution which optimises the objective function is called optimal feasible solution.

Convex set: Convex set is a set of points in a plane, on which the line segment joining any two points in the set, completely lies in the set.

Convex polygon: A polygon is a convex polygon if the line segments joining any two points inside it lies completely inside the polygon.

Theorem 1: Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

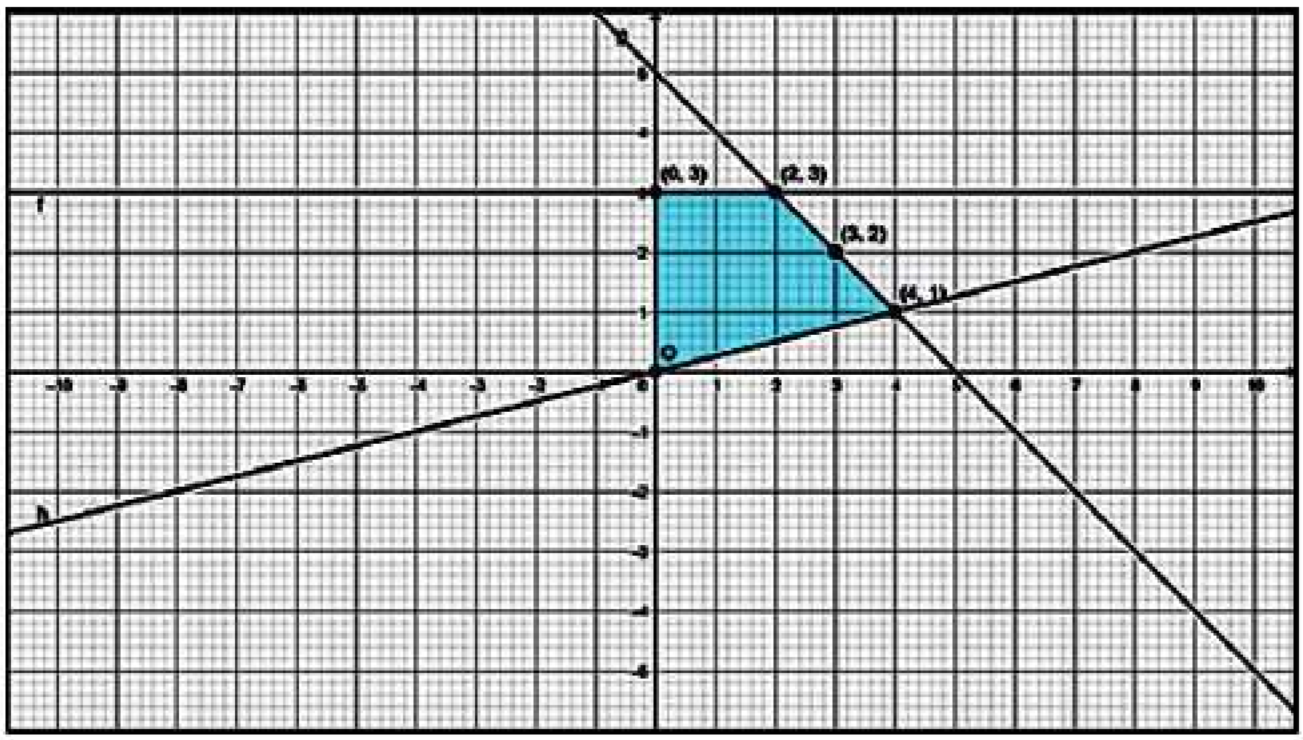
Remark: If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R.

(By Theorem 1).

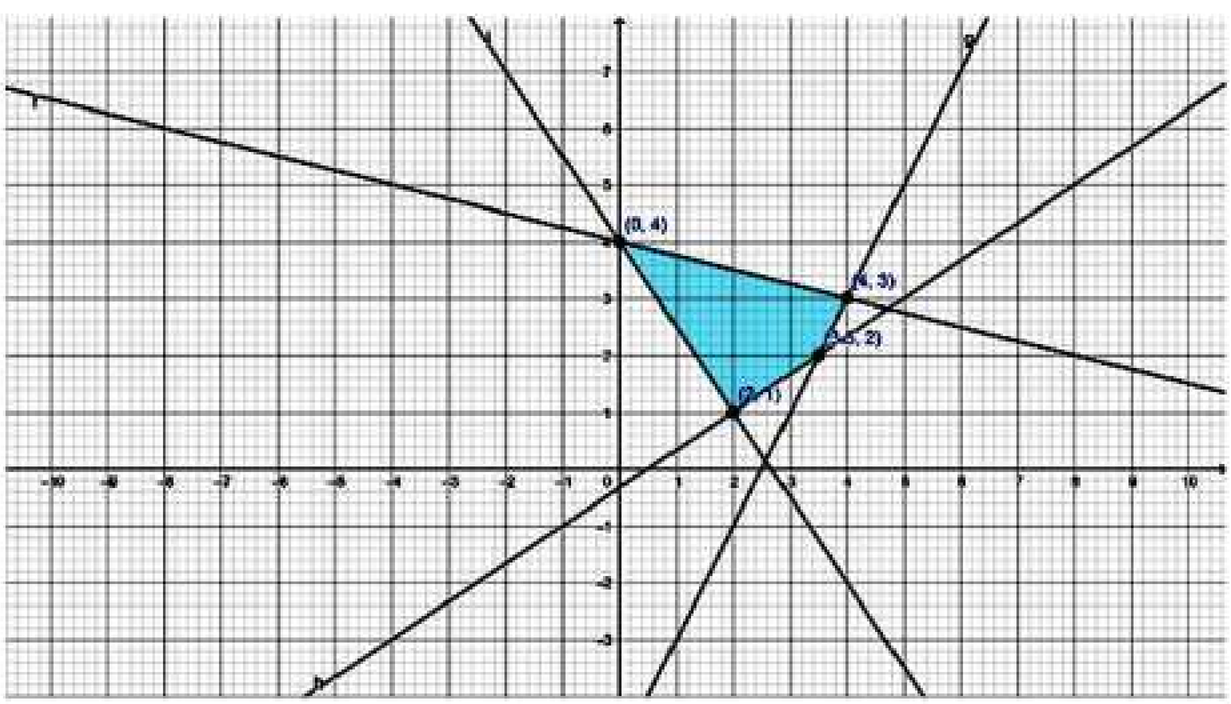
QUESTIONS:

1. If the objective function for an LPP is $Z = 5x + 7y$ and the corner points of the bounded feasible region are (0, 0), (5, 0), (2, 5) and (6, 1), then find the points where value of Z is maximum.
2. In an LPP with the objective function $Z = px + qy$ has same maximum value on two corner points of the feasible region, then find the number of points at which value of Z is maximum.
3. For an objective function $Z = ax + by$, $a, b > 0$, find the relationship between a and b when maximum value of Z occurs at (10, 5) and (0, 15).
4. If the objective function for an LPP is $Z = 2x + 3y$ and the corner points for bounded feasible region are (7, 1), (5, 2), (2, 5) and (0, 8), find the points on which we get minimum value of Z.

5. **The feasible region for an LPP is shown in the figure with objective function $Z = 5x+7y$. Find the point(s) at which Z is maximum.**



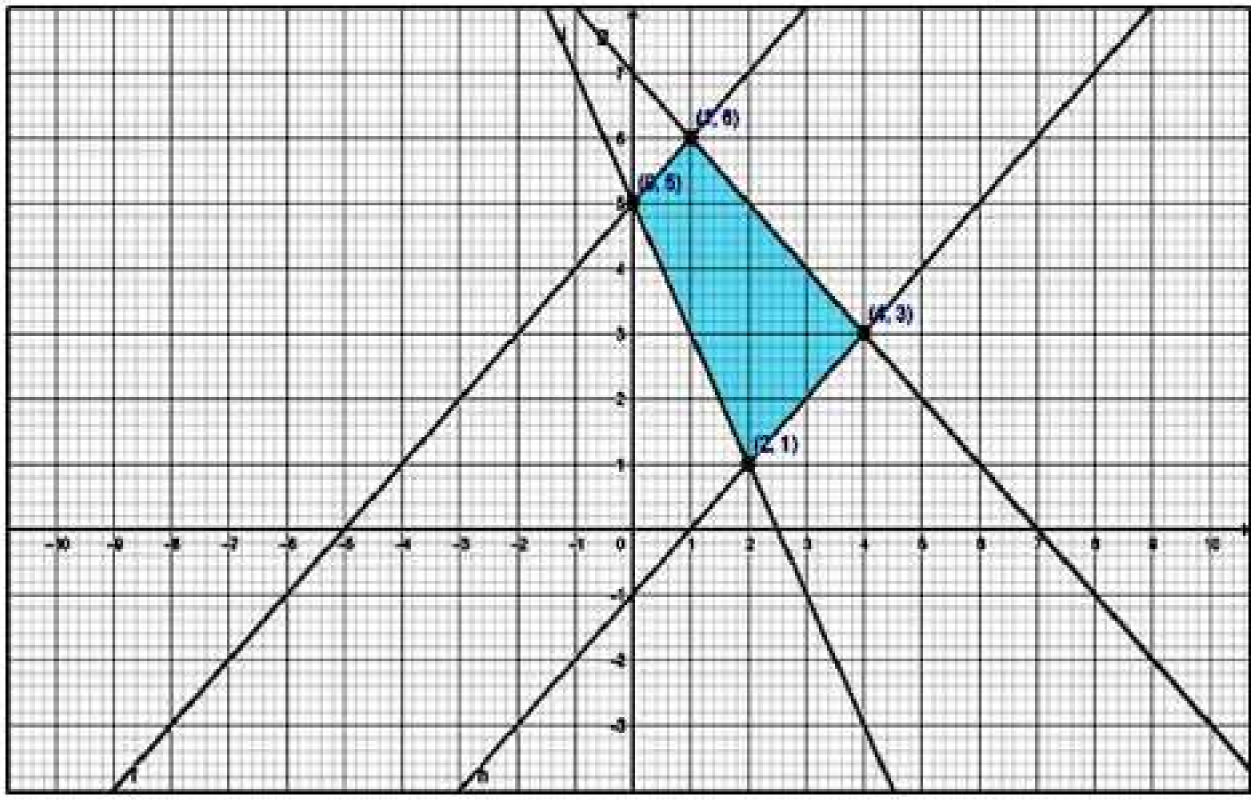
6. **The feasible region for an LPP is shown in the figure with objective function $Z = 5x + 4y$. Find the point(s) at which Z is minimum.**



7. Find the maximum value of $Z = 2x + 5y$ subject to constraints $x + y \leq 5$, $x, y \geq 0$.

8. Find the maximum value of $Z = 3x + 5y$ subject to constraints $3x - y \geq 0$, $x \leq 2$ and $x, y \geq 0$.

9. The feasible region for an LPP with objective function $Z = 3x + 5y$ is shown in the figure. Find the points at which Z is maximum.



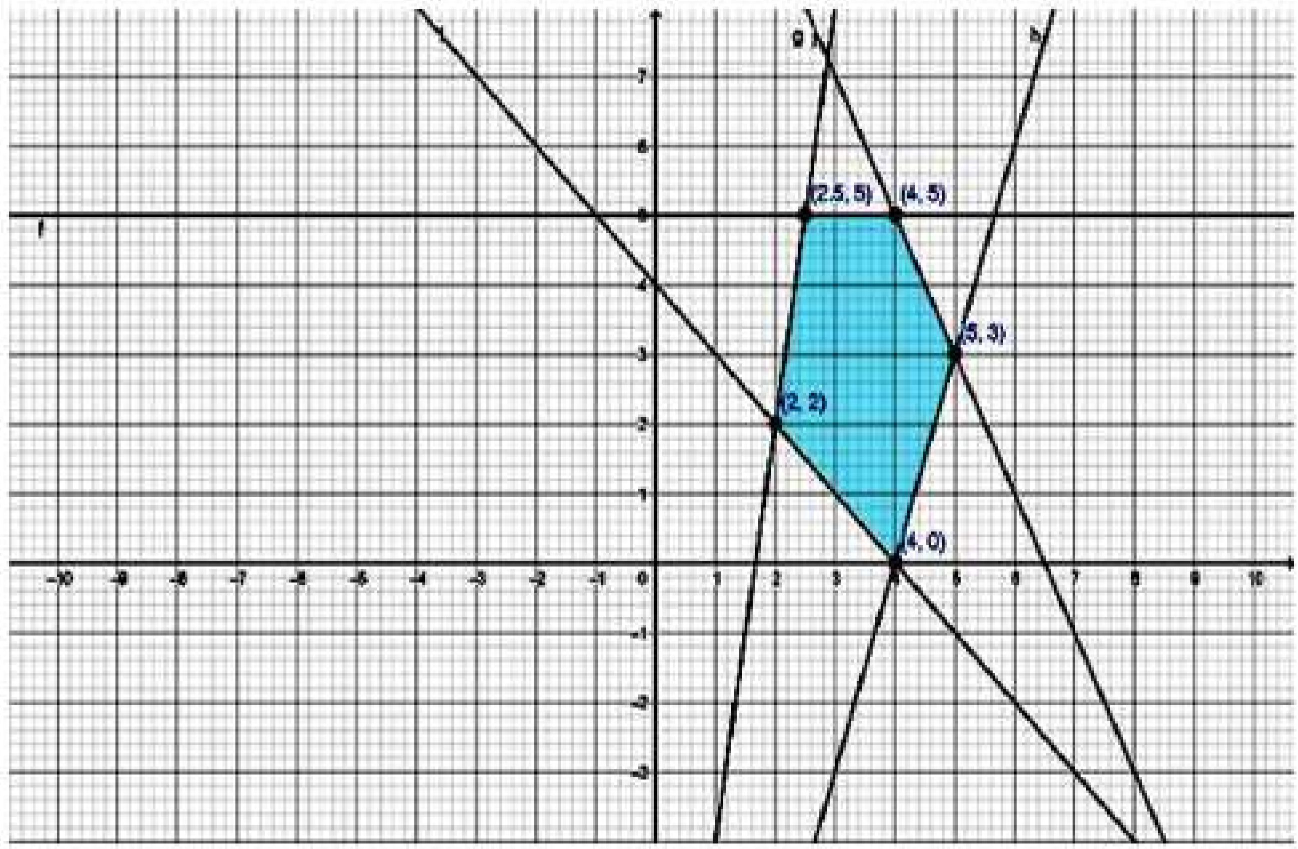
10. $Z = 7x + y$ subject to the constraints $2x + y \leq 6$, $x - y \geq 0$ and $x, y \geq 0$. Find the point(s) at which minimum value of Z occurs.

11. The corner points of the feasible reason for an LPP are (0, 3), (3, 0), (6, 2), (1, 2) and (0, 5). Let $Z = 4x + 5y$ be the objective function. Find the point(s) at which minimum value of Z occurs.

12. The corner points of the feasible region determined by the system of linear constraints are (1, 1), (0, 2), (3, 3) & (3, 0). Let $Z = px + qy$, $p,q>0$. Find the condition on p and q such that the minimum value of Z occurs at (3, 0) & (1, 1).

13. The corner points of the feasible region determined by the system of linear constraints are $(\frac{3}{2}, \frac{1}{4})$, (0, 0), (5, 0), (3, 2) & (0, 4). Let $Z = px + qy$, $p, q > 0$. Find the condition on p and q so that the maximum value of Z occurs at (3, 2) & (0, 4).

14. The feasible region for an LPP is shown in the figure. Let Z = 5x − 2y be the objective function. Find the point at which we get minimum value of Z.



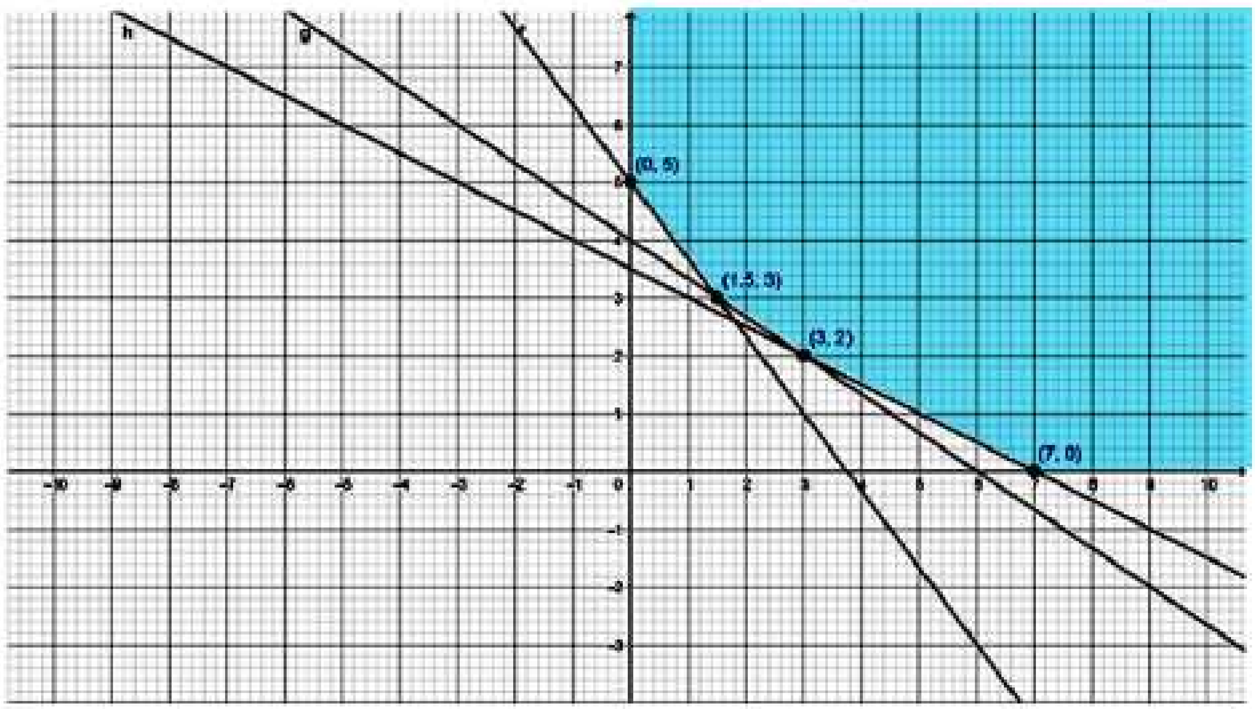
15. The corner points of the feasible region for an LPP are (0, 0), (0, 6), (2, 7), (5, 3) and (6, 0). Let P = 3 x + 2y be the objective function. Find the points at which we get maximum value of P.

16. Find the point(s) at which we get maximum value of Z = 3 x + 4y subject to the constraints $x + y \leq 20$, $2x - y \leq 40$ and $x, y \geq 0$.

17. Let Z = 5x + 2y be the objective function. The corner points of the feasible region are A $(\frac{3}{13}, \frac{24}{13})$, B $(\frac{3}{2}, \frac{15}{4})$, C $(\frac{7}{2}, \frac{3}{4})$ and D $(\frac{18}{7}, \frac{2}{7})$. Find the point(s) at which we get the maximum value of Z.

18. Let Z = 2x + 5y be the objective function. The corner points of the feasible region are A $(\frac{7}{5}, \frac{18}{5})$, B (5,3), C(3,4), D(4,2) and E (2,4). Find the point(s) at which we get the minimum value of Z.

19. The corner points of the unbounded region for an LPP are shown in the figure. Find the points where we get minimum value of the objective function $Z = 4x + 6y$.



20. For the objective function $Z = 5x + 2y$, the corner points of the bounded feasible region are (15, 5), (20, 3), (16, 10), (18, 12) & (12, 12). Find the minimum value of Z.

21. How many of the following points satisfy the inequality $3x - 2y > 5$?
(0, 0), (1, 1), (-1, 1), (1, -1), (-1, -1), (-2, 1), (2, -1), (-1, 2), & (-2, 1).

22. For the objective function LPP, $Z = 2x + 5y$, the coordinates of the corner points of the bounded feasible region are (0, 10), (9, 5), (2, 7), (16, 2) & (17, 5). Find the minimum value of Z.

23. The corner points of the feasible region determined by the system of linear constraints are (0, 20), (0, 9), (0, 50), (15, 15) & (9, 20). Suppose $Z = px + 3y$ where $p > 0$. If Z attains its maximum value at both the points (30, 30) and (0, 50), then find the value of p.

Fill in the blanks:

24. A solution of LPP which satisfies the non-negativity restrictions of the problem is called its _______________ solution.

25. The linear inequalities or equations or restrictions on the variables of a linear programming problem are called_________________.

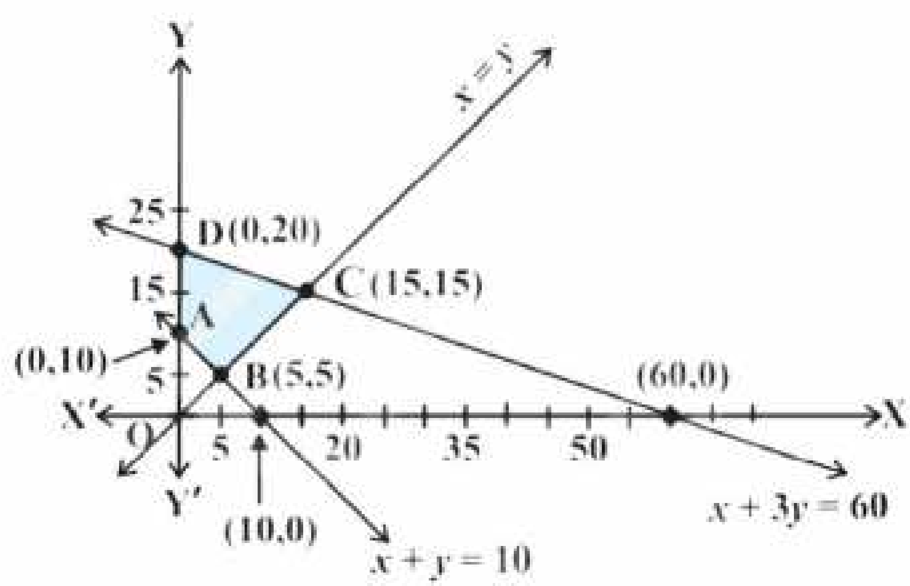
26. If the value of objective function Z can be increased or decreased indefinitely, such solution is called __________________.

27. The common region determined by all the constraints of an LPP is called the ___________ region.

28. The corner points of the feasible region determined by the system of linear constraints are (0, 0), (0, 40), (20, 40), (60, 20), (60, 0). If the objective function is $Z = x + y$ then find the maximum value of Z.

29. Corner points of the feasible region determined by the system of linear constraints are (0, 3), (1, 1) and (3, 0). Let $Z = px+qy$, $p, q > 0$ be the objective function. If Z attains its minimum value at (3, 0) & (1,1) then find the value of $\frac{q}{p}$.

30. It is given that shaded region ABCD as the feasible region determined by the system of linear constraints. If the objective Function be $Z = 2x + y$ then find the value of $Z_{Maximum} - Z_{Minimum}$.



ANSWERS

Q. No.	Answer	Q. No.	Answer
1	2, 5)	16	(0, 20)
2	Infinite	17	$(\frac{7}{2}, \frac{3}{4})$
3	a = b	18	(4, 2)
4	(5, 2)	19	At every point on the line segment joining the points (1.5, 3) and (3,2)
5	(2, 3)	20	84
6	(2, 1)	21	1
7	25	22	39
8	36	23	2
9	(1, 6)	24	feasible
10	(0, 0)	25	constraints
11	(3, 0)	26	unbounded solution
12	2p = q	27	feasible
13	2q = 3p	28	80
14	(2.5, 5)	29	2
15	(5, 3)	30	35

CHAPTER 13

PROBABILITY

POINTS TO REMEMBER:

1. <u>Random Experiment:-</u>

If in each trial of an experiment conducted under identical conditions, the outcome is not always the same, but may be any of the possible outcomes, then such an experiment is called random experiment.

Examples:

Tossing a fair coin, rolling an unbiased die, drawing a card from well shuffled pack of cards are all examples of random experiments.

2. <u>Sample Space:-</u>

The set of all possible outcomes in a random experiment is called a sample space.

Examples:

(i) In tossing a fair coin, we have: $S = \{H,T\}$

(ii) In a throw of a die, we have: $S = \{1,2,3,4,5,6\}$

(iii) When two coins are tossed together, $S = \{HT,TH,HH,TT\}$

(iv) When two dice are thrown together, S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2, 2), (2,3), (2,4), (2,5), (2,6),(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),(5,1),(5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6, 2), (6,3), (6,4), (6,5), (6,6)}

3. <u>Event</u>:- Any subset of sample space is called an event. It is denoted by E.

4. <u>Simple Event</u>:- Event with only one outcome is called a simple event.

5. Mutually Exclusive Events:-

A set of events is called mutually exclusive, if the happening of one event excludes the happening of the other. Thus E_1 and E_2 are mutually exclusive, if $E_1 \cap E_2 = \emptyset$.

Example: In a throw of a die, we have $S = \{1,2,3,4,5,6\}$

Let E_1 = event of getting a number less than 3 = {1, 2}.

And, E_2 = event of getting a number greater than 4 ={5,6}. Clearly, $E_1 \cap E_2 = \phi$.

6. Exhaustive Events:-

The events E_1, E_2 , ..., E_k such that $E_1 \cup E_2 \cup .... \cup E_k = S$ are called exhaustive events.

7. Probability:-

In a random experiment, let S be the sample space and let $E \subseteq S$ where E is an event.

$$P(E) = \frac{number\ of\ distinct\ elements\ in\ E}{number\ of\ distinct\ elements\ in\ S} = \frac{n(E)}{n(S)}$$

8. Complementary Events:-

Let S be the sample space and Let $E \subseteq S$. Then, an event containing those outcomes which are in S but not in E is called complementary event of E and is denoted by E^c , $\overline{E}$ or E'.

9. Addition Theorem:-

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Remarks, If $E_1 \cap E_2 = \phi$, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

10. Independent Events:-

Two events are said to be independent, if the occurrence of one does not depends upon the occurrence of other.

Example: Suppose two fair coins are tossed.

Let E_1 = event of getting a tail on first coin and E_2 = event of getting a tail on second coin.

Clearly, the occurrence of tail on second coin does not depend upon the occurrence of tail on first coin.

So, E_1 and E_2 are independent events.

11. <u>Multiplication Theorem</u>:-

If E_1 and E_2 are independent events, then $P(E_1 \cap E_2) = P(E_1) \times P(E_2)$

12. <u>Conditional Probability</u>:-

The probability of the occurrence of an event E_1, when an event E_2 has already occurred is called the conditional probability and is denoted by $P(E_1 / E_2)$.

We have: (i) $P(E_1 / E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$ (ii) $P(E_2 / E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$

13. <u>Theorem of Total Probability</u>: Let events $E_1, E_2, \ldots, E_k$ form partitions of the sample space S, where all the events have a non-zero probability of occurrence. For any event A associated with S, according to the total probability theorem,

$$P(A) = P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + \cdots + P(E_k).P(A/E_k)$$

14. <u>Baye's Theorem</u>:-

If $E_1, E_2, \ldots, E_k$ forms a set of mutually exclusive and exhaustive events of a random experiment and A is an event, then

$$P(E_i/A) = \frac{P(E_i).P(A/E_i)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + \cdots + P(E_k).P(A/E_k)}$$

Specific case: for n = 2,

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

15. Binomial Theorem of Probability:-

Suppose there are n independent trails of an experiment with p as the probability of success and $q = (1-p)$ as the probability of failure. Then,

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, $r = 0,1,2\ldots,n$

16. More Results:-

(i) $P(\bar{E}) = 1 - P(E)$

(ii) $P(E - F) = P(E) - P(E \cap F)$

(iii) $P(\phi) = 0$, $P(S) = 1$ and $0 \le P(E) \le 1$.

17. Binomial Distribution:-

Suppose n trials are made in an experiment. Let p = probability of getting a success and q = probability of getting a failure. Then, clearly $p + q = 1$.

(i) Binomial Distribution is $(p + q)^n$.

(ii) Mean = np

(iii) Variance = npq

(iv) Standard Deviation = $\sqrt{npq}$

QUESTIONS:

1. A pair of dice is thrown simultaneously and the numbers appearing on them have sum greater than or equal to 10, then what is the probability of getting a sum of 10?

2. A bag contains 4 white and 2 black balls. Another bag contains 3 white and 5 black balls. If one ball is drawn at random from each bag, then what is the probability that both balls drawn are white?

3. A bag contains 5 white and 4 red balls. Another bag contains 4 white and 2 red balls. If one ball is drawn at random from each bag, then what is the probability that one ball is white and other is red?

4. If E and F are mutually exclusive events, then find $P(E \cap F)$.

5. If E and F are mutually exclusive events with $P(E) = 0.45$ & $P(F) = 0.35$, then find $P(E \cup F)$.

6. For any two events E and F such that $P(E) = 0.65$, $P(F) = 0.55$ & $P(E \cap F) = 0.3$, then find $P(E - F)$.

7. The probabilities of occurrence of two events E and F are 0.25 & 0.50 respectively. The probability of their simultaneous occurrence is 0.14. What is the probability that neither E occurs nor F occurs?

8. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\bar{A}) + P(\bar{B})$.

9. Let E and F be events such that P(E) = 1/3, P(F) = 1/4 and $P(E \cap F) = 1/5$, then find $P(E / F)$.

10. Let E and F be events such that P(E) = 1/4, P(F) = 1/3 and $P(E \cap F) = 1/5$, then find $P(\bar{F} / \bar{E})$.

11. A die is rolled once. If the outcome is an odd number, then what is the probability that it is prime?

12. If E and F are events such that $P(E) = 0.4$, $P(F) = 0.8$ and P (F/E) = 0.6 , then find P(E/F).

13. A die is thrown twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 2 has appeared at least once?

14. In a class 40% of the students study mathematics, 25% study biology and 15% study both. One student is selected at random, what is the probability that the student studies mathematics, if it is known that he studies biology?

15. A couple has 2 children. What is the probability that both are boys, if it is known that at least one of them is a boy?

16. Two numbers are selected at random from the integers 1 to 9. If the sum is even, then what is the probability that both the numbers are odd?

17. If E and F are independent events such that $P(E) = 0.65$, then find $P(\overline{E} / \overline{F})$.

18. If E and F are independent events with $P(E) = 0.5$ & $P(F) = 0.25$,then what is the probability of simultaneous occurrence of both?

19. A can solve 90% of the problems given in a book while B can solve 70%. A problem is selected at random from the book , then what is the probability that at least one of them will solve it?

20. The probability of a problem being solved by two students independently are 1/3 and ½ respectively. What is the probability that the problem is solved?

21. The probabilities of solving a problem by three students A, B and C are 1/2, 1/3 and 1/4 respectively. What is the probability that the problem is solved?

22. A speaks truth in 75% of the cases and B in 80% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

23. If the probabilities that A and B will die within a year are p and q respectively, then what is the probability that only one of them will be alive at the end of the year?

24. An unbiased die is tossed twice. What is the probability of getting 4, 5 or 6 on the first toss and 1, 2, 3 or 4 on the second toss?

25. A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is 1/7 and the probability of wife's selection is 1/5. What is the probability that only one of them is selected?

26. If A and B are independent events such that $P(\overline{A}) = 0.65$, $P(A \cup B) = 0.65$ and $P(B) = p$, then what is the value of p ?

27. A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is drawn from each bag. What is the probability that one ball is red and another one is black?

28. A coin is tossed 5 times. What is the probability that tail appears an odd number of times?

29. A pair of dice is thrown 7 times. If getting a total of 7 is considered a success, then what is the probability of getting atmost 6 successes?

30. A die is thrown twice and the sum of the numbers appearing is noted to be 6. Find the conditional probability that the number 4 has appeared at least once.

31. Given that the events A and B are such that P(A) = 1/2, P (A ∪ B) = 3/5 and P(B) = p. Find the value of p, if they are

(i) mutually exclusive

(ii) independent

32. 5 cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the probability that only 3 cards are spades.

33. A coin is tossed twice. If the outcome is at most one tail, then what is the probability that both head and tail have appeared?

34. The probability that a person A hits a target is 1/3 and that person B hits is 2/5. What is the probability that the target will be hit if both A and B shoot at it?

35. In a school there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen at random studies in class XII, given that the chosen student is a girl?

36. A die thrown three times. Events A and B are defined as

A: 4 on the third throw

B: 6 on the first and 5 on the second throw.

Find the probability of A given that B has already occurred.

37. Mother, father and son line up at random for a family picture. The events E and F are defined as :- E : Son on one end & F : Father in middle

Find P (E/F).

38. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

39. A Box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. What is the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale?

40. Three cards are drawn successively without replacement from a pack of 52 well shuffled cards. What is the probability of getting two kings and an ace?

41. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the values x, has the following form, where k is some unknown constant

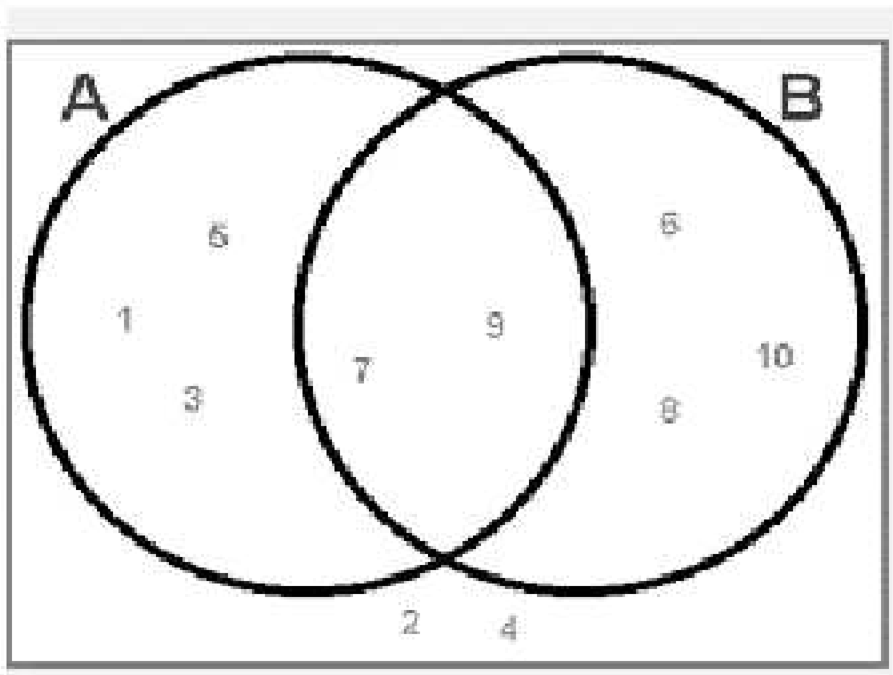
$$P(X=x)=\begin{cases} 0.1 : if\ x=0 \\ kx : if\ x= 1 or 2 \\ k(5-x): if\ x=3 or 4 \\ 0 : otherwise \end{cases}$$

Find the value of k.

42. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. What is the probability that it is actually a six?

43. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

44. What is the value of $\mathbf{P\left(\frac{A}{B}\right)+P\left(\frac{B}{A}\right)}$.



45. If a leap year is selected at random, what is the chance that it will contain 53 Tuesdays?

46. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, then find the conditional probability that both are girls, given that the youngest is a girl.

47. P speaks truth in 70% of the cases and Q in 80% of the cases. In what percent of cases are they likely to agree in stating the same fact?

48. A random variable X has following probability distribution:

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find the value of k.

49. Two fair dice are tossed simultaneously. Find the conditional probability of getting two sixes given that at least one six has occured?

50. If A and B are two mutually exclusive events, then what is the relation between P (A) and P (B)?

ANSWERS

Q. No.	Answer	Q. No.	Answer
1	1/2	26	6/13
2	1/4	27	21/40
3	13/27	28	1/2
4	0	29	$1-(1/6)^7$
5	4/5	30	2/5
6	7/20	31	(i)1/10 (ii) 1/5
7	39/100	32	45/512
8	6/5	33	2/3
9	4/5	34	3/5
10	37/45	35	1/10
11	2/3	36	1/6
12	3/10	37	1
13	1/3	38	5/9
14	3/5	39	44/91
15	1/3	40	6/(13x17x 25)
16	5/8	41	3/20
17	7/20	42	3/8
18	1/8	43	8/11
19	0.97 or 97%	44	4/5
20	2/3	45	2/7
21	3/4	46	1/2
22	35%	47	62%
23	$p+q-2pq$	48	1/10
24	1/3	49	1/11
25	2/7	50	$P(A)\leq P(\bar{B})$ or $P(B)\leq P(\bar{A})$

MENTAL MATHS

DIRECTORATE OF EDUCATION GOVT.OF N.C.T.OF DELHI

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