

# **DIRECTORATE OF EDUCATION**

**GNCT of Delhi**

## **SUPPORT MATERIAL**

**(2020-2021)**

**Class : XII**

## **MATHEMATICS**

Under the Guidance of

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### **MESSAGE**

The importance of adequate practice during examinations can never be overemphasized. I am happy that support material for classes IX to XII has been developed by the Examination Branch of Directorate of Education. This material is the result of immense hard work, co-ordination and cooperation of teachers and group leaders of various schools. The purpose of the support material is to impart ample practice to the students for preparation of examinations. It will enable the students to think analytically & rationally, and test their own capabilities and level of preparation.

The material is based on latest syllabus prepared by the NCERT and adopted by the CBSE for the academic session 2020-21 and covers different levels of difficulty. I expect that Heads of Schools and Teachers will enable and motivate students to utilize this material during zero periods, extra classes and regular classes best to their advantage.

I would like to compliment the team of Examination Branch for their diligent efforts of which made it possible to accomplish this work in time. I also take this opportunity to convey my best wishes to all the students for success in their endeavours.

(Manisha Saxena)

**BINAY BHUSHAN, IAS**



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Dear Students,

Directorate of Education is committed to providing qualitative and best education to all its students. The Directorate is continuously engaged in the endeavor to make available the best study material for uplifting the standard of its students and schools.

Every year, the expert faculty of Directorate reviews and updates Support Material. The expert faculty of different subjects incorporates the changes in the material as per the latest amendments made by CBSE to make its students familiar with new approaches and methods so that students do well in the examination.

The book in your hand is the outcome of continuous and consistent efforts of senior teachers of the Directorate. They have prepared and developed this material especially for you. A huge amount of money and time has been spent on it in order to make you updated for annual examination.

Last, but not the least, this is the perfect time for you to build the foundation of your future. I have full faith in you and the capabilities of your teachers. Please make the fullest and best use of this Support Material.

  
**BINAY BHUSHAN**  
**DIRECTOR (EDUCATION)**



**Dr. (Mrs.) Saroj Bala Sain**

Addl. Director of Education  
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सत्यमेव जयते

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Date : 03-10-2019

I am very much pleased to forward the Support Material for classes IX to XII. Every year, the Support Material of most of the subjects is updated/revised as per the most recent changes made by CBSE. The team of subject experts, officers of Exam Branch, members of Core Academic Unit and teachers from various schools of Directorate has made it possible to make available unsurpassed material to students.

Consistence use of Support Material by the students and teachers will make the year long journey seamless and enjoyable. The main purpose to provide the Support Material for the students of government schools of Directorate is not only to help them to avoid purchasing of expensive material available in the market but also to keep them updated and well prepared for exam. The Support Material has always been a ready to use material, which is matchless and most appropriate.

I would like to congratulate all the Team Members for their tireless, unremitting and valuable contributions and wish all the best to teachers and students.

(Dr. Saroj Bala Sain)  
Addl.DE (School/Exam)



**DIRECTORATE OF EDUCATION**  
GNCT of Delhi

**SUPPORT MATERIAL**  
**(2020-2021)**

**MATHEMATICS**

**Class : XII**

**NOT FOR SALE**

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## भारत का संविधान

भाग 4क

### नागरिकों के मूल कर्तव्य

#### अनुच्छेद 51क

**मूल कर्तव्य** — भारत के प्रत्येक नागरिक का यह कर्तव्य होगा कि वह —

1. संविधान का पालन करे और उसके आदर्शों, संस्थाओं, राष्ट्र ध्वज और राष्ट्रगान का आदर करें।
2. स्वतंत्रता के लिए हमारे राष्ट्रीय आंदोलन को प्रेरित करने वाले उच्च आदर्शों को हृदय में संजोए रखे और उनका पालन करे।
3. भारत की प्रभुता, एकता और अखंडता की रक्षा करे और उसे अक्षुण्ण रखे।
4. देश की रक्षा करे।
5. भारत के सभी लोगों में समरसता और समान भ्रातृत्व की भावना का निर्माण करे।
6. हमारी सामाजिक संस्कृति की गौरवशाली परंपरा का महत्त्व समझे और उसका निर्माण करे।
7. प्राकृतिक पर्यावरण की रक्षा और उसका संवर्धन करे।
8. वैज्ञानिक दृष्टिकोण और ज्ञानार्जन की भावना का विकास करे।
9. सार्वजनिक संपत्ति को सुरक्षित रखे।
10. व्यक्तिगत एवं सामूहिक गतिविधियों के सभी क्षेत्रों में उत्कर्ष की ओर बढ़ने का सतत प्रयास करे।
11. माता—पिता या संरक्षक द्वारा 6 से 14 वर्ष के बच्चों हेतु प्राथमिक शिक्षा प्रदान करना (86वां संशोधन)।

# CONSTITUTION OF INDIA

## Part IV A (Article 51 A)

### Fundamental Duties

**Fundamental Duties :** It shall be the duty of every citizen of India —

1. to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;
2. to cherish and follow the noble ideals which inspired our national struggle for freedom;
3. to uphold and protect the sovereignty, unity and integrity of India;
4. to defend the country and render national service when called upon to do so;
5. to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;
6. to value and preserve the rich heritage of our composite culture;
7. to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures.
8. to develop the scientific temper, humanism and the spirit of inquiry and reform;
9. to safeguard public property and to adjure violence;
10. to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement.
11. who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.



## भारत का संविधान

### उद्देशिका

हम, भारत के लोग, भारत को एक (सम्पूर्ण प्रभुत्व—सम्पन्न समाजवादी पंथनिरपेक्ष लोकतंत्रात्मक गणराज्य) बनाने के लिए, तथा उसके समस्त नागरिकों को :

सामाजिक, आर्थिक और राजनैतिक न्याय,

विचार, अभिव्यक्ति, विश्वास, धर्म

और उपासना की स्वतंत्रता,

प्रतिष्ठा और अवसर की समता

प्राप्त करने के लिए,

तथा उन सब में,

व्यक्ति की गरिमा और (राष्ट्र की एकता

और अखंडता) सुनिश्चित करने वाली बंधुता

बढ़ाने के लिए

हम दृढ़संकल्प होकर इस संविधान को आत्मार्पित करते हैं।

# THE CONSTITUTION OF INDIA

## PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a **(SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC)** and to secure to all its citizens :

**JUSTICE**, social, economic and political,

**LIBERTY** of thought, expression, belief, faith and worship,

**EQUALITY** of status and of opportunity; and to promote among them all

**FRATERNITY** assuring the dignity of the individual and the **(unity an integrity of the Nation)**;

WE DO HEREBY GIVE TO OURSELVES THIS CONSTITUTION.

# MATHEMATICS CLASS 12

## SYLLABUS

### Course Structure

Unit	Topic	Marks
I.	Relations and Functions	08
II.	Algebra	10
III.	Calculus	35
IV.	Vectors and 3-D Geometry	14
V.	Linear Programming	05
VI.	Probability	08
<b>Total :</b>		<b>80</b>
<b>INTERNAL ASSESSMENT</b>		<b>20</b>



## CHAPTER 1

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### RELATIONS AND FUNCTIONS

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#### IMPORTANT POINTS TO REMEMBER

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- Relation  $R$  from a set  $A$  to a set  $B$  is subset of  $A \times B$  and Relation  $R$  in set  $A$  is a subset of  $A \times A$ .
- If  $n(A) = r$ ,  $n(B) = s$  from set  $A$  to set  $B$  then  $n(A \times B) = rs$  and number of relations  $= 2^{rs}$
- $\emptyset$  is also a relation defined on set  $A$ , called the void (empty) relation.
- $R = A \times A$  is called universal relation.
- **Reflexive Relation:** Relation  $R$  defined on set  $A$  is said to be reflexive if  $(a, a) \in R \forall a \in A$ .
- **Symmetric Relation :** Relation  $R$  defined on set  $A$  is said to be symmetric iff  $(a, b) \in R \Rightarrow (b, a) \in R \forall a, b \in A$
- **Transitive Relation :** Relation  $R$  defined on set  $A$  is said to be transitive if  $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R \forall a, b, c \in A$
- **Equivalence Relation:** A relation defined on set  $A$  is said to be equivalence relation if it is reflexive, symmetric and transitive.
- **Equivalence class of an element:** Let  $R$  be an equivalence relation of set  $A$ , then equivalence class of  $a \in A$  is  $[a] = \{ b \in A : (b, a) \in R \}$ .
- **One-One Function :**  $f : A \rightarrow B$  is said to be one-one if distinct elements in  $A$  have distinct images in  $B$ . i.e.  $\forall x_1, x_2 \in A$  such that  $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ .

OR

$$\forall x_1, x_2 \in A \text{ such that } f(x_1) = f(x_2)$$

$$\Rightarrow x_1 = x_2$$

One-one function is also called injective function.

- **Onto function (surjective):** A function  $f : A \rightarrow B$  is said to be onto iff  $R_f = B$  i.e.  $\forall b \in B$ , there exists  $a \in A$  such that  $f(a) = b$
- **Bijjective Function :** A function which is both injective and surjective is called bijective function.
- **Composition of Two Functions :** If  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  are two functions, then composition of  $f$  and  $g$  denoted by  $g \circ f$  is a function from  $A$  to  $C$  given by,  $(g \circ f)(x) = g(f(x)) \forall x \in A$   
Clearly  $g \circ f$  is defined if Range of  $f$  is a subset of domain of  $g$ . Similarly  $f \circ g$  can be defined.
- **Invertible Function:** A function  $f : X \rightarrow Y$  is invertible iff it is bijective.  
If  $f : X \rightarrow Y$  is bijective function, then function  $g : Y \rightarrow X$  is said to be inverse of  $f$  iff  $f \circ g = I_Y$  and  $g \circ f = I_X$   
when  $I_X, I_Y$  are identity functions.
- Inverse of  $f$  is denoted by  $f^{-1}$ . [  $f^{-1}$  does not mean  $\frac{1}{f}$  ]
- Let  $A$  and  $B$  are two non empty set that  $n(A) = p$  and  $n(B) = q$   
Then
  - a) Number of functions from  $A$  to  $B = q^p$
  - b) Number of one-one functions from  $A$  to  $B = \begin{cases} {}^qP_p & , p \leq q \\ 0 & , p > q \end{cases}$
  - c) Number of onto function from  $A$  to  $B = \begin{cases} \sum_{r=1}^q (-1)^{q-r} {}^qC_r r^p & , p \geq q \\ 0 & , p < q. \end{cases}$
  - d) Number of bijective functions from  $A$  to  $B = \begin{cases} p! & , p = q \\ 0 & , p \neq q. \end{cases}$

## ONE MARK QUESTIONS

1. If  $A$  is the set of students of a school then write, which of following relations are Universal, Empty or neither of the two.

$$R_1 = \{(a, b) : a, b \text{ are ages of students and } |a - b| > 0\}$$

$$R_2 = \{(a, b) : a, b \text{ are weights of students, and } |a - b| < 0\}$$

$$R_3 = \{(a, b) : a, b \text{ are students studying in same class}\}$$

2. Is the relation  $R$  in the set  $A = \{1, 2, 3, 4, 5\}$  defined as

$$R = \{(a, b) : b = a + 1\} \text{ reflexive?}$$

3. If  $R$ , is a relation in set  $N$  given by

$$R = \{(a, b) : a = b - 3, b > 5\},$$

then does element  $(5, 7) \in R$ ?

4. If  $f : \{1, 3\} \rightarrow \{1, 2, 5\}$  and  $g : \{1, 2, 5\} \rightarrow \{1, 2, 3, 4\}$  be given by  $f = \{(1, 2), (3, 5)\}$ ,  $g = \{(1, 3), (2, 3), (5, 1)\}$ , write  $g \circ f$ .

5. Let  $g, f : R \rightarrow R$  be defined by

$$g(x) = \frac{x+2}{3}, f(x) = 3x - 2. \text{ write } f \circ g(x)$$

6. If  $f : R \rightarrow R$  defined by

$$f(x) = \frac{2x-1}{5}$$

be an invertible function, write  $f^{-1}(x)$ .

7. If  $f(x) = \log x$  and  $g(x) = e^x$ . Find  $f \circ g$  and  $g \circ f$ ,  $x > 0$ .

8. If  $n(A) = n(B) = 3$ , then how many bijective functions from  $A$  to  $B$  can be formed?
9. Is  $f:N \rightarrow N$  given by  $f(x) = x^2$ , one-one? Give reason.
10. If  $f:R \rightarrow A$ , given by  
 $f(x) = x^2 - 2x + 2$  is onto function, find set  $A$ .
11. If  $f:A \rightarrow B$  is bijective function such that  $n(A) = 10$ , then  $n(B) = ?$
12. If  $f:R \rightarrow R$  defined by  $f(x) = \frac{x-1}{2}$ , find  $(f \circ f)(x)$
13.  $R = \{(a, b) : a, b \in N, a \neq b \text{ and } a \text{ divides } b\}$ . Is  $R$  reflexive? Give reason
14. Is  $f:R \rightarrow R$ , given by  $f(x) = |x - 1|$  one-one? Give reason
15.  $f:R \rightarrow B$  given by  $f(x) = \sin x$  is onto function, then write set  $B$ .
16. If  $f(x) = \log\left(\frac{1+x}{1-x}\right)$ , show that  $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$ .
17. State the reason for the relation  $R$  in the set  $\{1,2,3\}$  given by  $R = \{(1,2), (2,1)\}$  not to be transitive.
18. If  $R = \{(x,y) : x+2y = 8\}$  is a relation on  $N$ , write the range of  $R$ .
19. Let  $A = \{0,1,2,3\}$  and define a relation  $R$  on  $A$  as follows:  
 $R = \{(0,0) (0,1) (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$ , Is  $R$  reflexive?  
Symmetric? transitive?
20. Consider the set  $A = \{1,2,3\}$ . Write the smallest equivalence relation  $R$  on  $A$ .



21. If the relation  $R$  defined in  $N$  by  $aRb$  if  $2a+3b=30$ .  
Then the set  $R$ .
22. Let  $f: R \rightarrow R$  be defined by  $f(x) = \frac{x}{\sqrt{1+x^2}}$  find  $(f \circ f \circ f)(x)$
23. Let  $f(x) = \{4 - (x-7)^3\}$ , find  $f^{-1}(x)$ .
24. Let  $A = \{1, 2, 3\}$  and  $B = \{4, 5, 6, 7\}$  and let  $f = \{(1, 4), (2, 5), (3, 6)\}$  be a function from  $A$  to  $B$ . State whether  $f$  is one-one or not.
25. If  $A = \{1, 2, 3, 4\}$  and  $B = \{-1, 3\}$ , then what is the number of onto functions from  $A$  to  $B$ ?
26. If  $A = \{-1, 2, 3\}$  and  $B = \{0, 3, 5\}$  then what is the number of bijections from  $A$  to  $B$ ?
27. If  $A = \{-1, 2, 3\}$  and  $B = \{0, 3, 5, 7\}$  then what is the number of bijections from  $A$  to  $B$ ?

### TWO MARK QUESTIONS

1. Check the following functions for one-one and onto
  - (a)  $f: R \rightarrow R, f(x) = \frac{2x-3}{7}$
  - (b)  $f: R \rightarrow R, f(x) = |x + 1|$
  - (c)  $f: R - \{2\} \rightarrow R, f(x) = \frac{3x-1}{x-2}$
  - (d)  $f: R - \{-1, 1\}, f(x) = \sin^2 x$ .
2. If  $f, g: R \rightarrow R$  be two functions defined by  $f(x) = |x| + x$  and  $g(x) = |x| - x$ , find  $g \circ f$  and  $f \circ g$ .
3. If  $f: [1, \alpha] \rightarrow [2, \alpha]$  is defined by  $f(x) = x + \frac{1}{x}$ , find  $f^{-1}(x)$
4. Let  $A = \{1, 2, 3\}$  and define  $R = \{(a, b) : a - b = 12\}$ . Show that  $R$  is empty relation on Set  $A$ .
5. Let  $A = \{1, 2, 3\}$  and define  $R = \{(a, b) : a + b > 0\}$ . Show that  $R$  is a universal relation on set  $A$ .
6. Let  $A = \{a, b, c\}$ . How many relation can be defined in the set? How many of these are reflexive?
7. Let  $f: R \rightarrow R$  be defined by  $f(x) = x^2 + 1$ , find the pre image of 17 and -3
8. If  $f: R \rightarrow R, g: R \rightarrow R$ , given by  $f(x) = [x], g(x) = |x|$ , then find  $f \circ g \left( -\frac{2}{3} \right)$   
and  $g \circ f \left( -\frac{2}{3} \right)$ .

9. Let  $Q$  be the set of rational number and  $R$  be the relation on  $Q$  defined by  $R=\{(x,y): x,y \in Q, x^2+3y^2=4xy\}$  check whether  $R$  is reflexive, symmetric and transitive.
10. Let  $A=\{2,4,6,8\}$  and  $R$  be the relation “is greater than” on the set  $A$ . Write  $R$  as a set of order pairs. Is this relation  
(i) reflexive? (ii) symmetric? (iii) equivalence relation?  
Justify your answer.
11. Let  $N$  be the set of natural numbers and relation  $R$  on  $N$  be defined by  $R=(x,y): x,y \in N, x+4y=10\}$  check whether  $R$  is reflexive, symmetric and transitive.

### FOUR MARK QUESTIONS

1. Let  $f : R - \left\{ \frac{-4}{3} \right\} \rightarrow R - \left\{ \frac{4}{3} \right\}$  be a function given by  $f(x) = \frac{4x}{3x+4}$ .  
Show that  $f$  is invertible with  $f^{-1}(x) = \frac{4x}{4-3x}$ .
2. Let  $R$  be the relation on set  $A = \{x : x \in \mathbb{Z}, 0 \leq x \leq 10\}$  given by  $R = \{(a, b) : (a - b) \text{ is divisible by } 4\}$ . Show that  $R$  is an equivalence relation. Also, write all elements related to 4.
3. Show that function  $f : A \rightarrow B$  defined as  $f(x) = \frac{3x+4}{5x-7}$  where  $A = R - \left\{ \frac{7}{5} \right\}$ ,  $B = R - \left\{ \frac{3}{5} \right\}$  is invertible and hence find  $f^{-1}$ .
4. Show that the relation  $R$  defined by  $(a, b) R(c, d) \Leftrightarrow a + d = b + c$  on the set  $N \times N$  is an equivalence relation.
5. Show that  $f : R_+ \rightarrow R_+$  defined by  $f(x) = \frac{1}{2x}$  is bijective, where  $R_+$  is the set of all non-zero positive real numbers.
6. Let  $A = \{1, 2, 3, \dots, 12\}$  and  $R$  be a relation in  $A \times A$  defined by  $(a, b) R(c, d)$  if  $ad = bc \forall (a, b), (c, d) \in A \times A$ . Prove that  $R$  is an equivalence relation. Also obtain the equivalence class  $[(3, 4)]$ .

7. Let  $A = \{1, 2, 3, \dots, 9\}$  and  $R$  be the relation defined on  $A \times A$  by  $(a, b) R (c, d)$  iff  $a + d = b + c$ . Prove that  $R$  is an equivalence relation. Also find the equivalence class  $[(2, 5)]$ .
8. Show that  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by 
$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$
 is both one-one and onto.
9. Consider  $f: \mathbb{R}_+ \rightarrow [4, \infty)$  given by  $f(x) = x^2 + 4$ . Show that  $f$  is invertible with the inverse ' $f^{-1}$ ' given by  $f^{-1}(y) = \sqrt{y - 4}$ , where  $\mathbb{R}_+$  is the set of non-negative real numbers.
10. Let  $A = \mathbb{R} - \{2\}$  and  $B = \mathbb{R} - \{1\}$  if  $f: A \rightarrow B$  is a function defined by  $f(x) = \frac{x-1}{x-2}$  show that  $f$  is one-one and onto. Hence find  $f^{-1}$ .
11. If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a function defined by  $f(x) = ax + b$  for all  $x \in \mathbb{R}$ , then find the constants  $a$  and  $b$  such that  $f \circ f = I_{\mathbb{R}}$ .
12. Prove that the relation  $R$  in the set  $A = \{5, 6, 7, 8, 9\}$ , given by  $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$  is an equivalence relation. Find all elements related to element 6.
13. Let  $f: \mathbb{W} \rightarrow \mathbb{W}$  be defined as  $f(n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$  show that  $f$  is invertible and find the inverse of  $f$ . Here  $\mathbb{W}$  is the set of whole numbers.

### SIX MARK QUESTIONS

1. Let  $\mathbb{N}$  denote the set of all natural numbers and  $R$  be the relation on  $\mathbb{N} \times \mathbb{N}$  defined by  $(a, b) R (c, d)$  if  $ad(b + c) = bc(a + d)$ . Show that  $R$  is an equivalence relation.
2. Let  $f: \mathbb{N} \rightarrow \mathbb{R}$  be a function defined as  $f(x) = 4x^2 + 12x + 15$ .  
Show that  $f: \mathbb{N} \rightarrow S$ , where  $S$  is the range of  $f$ , is invertible. Also find the inverse of  $f$ . Hence find  $f^{-1}(31)$ .
3. If the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = 2x - 3$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  by  $g(x) = x^3 + 5$ , then show that  $f \circ g$  is invertible. Also find  $(f \circ g)^{-1}(x)$ , hence find  $(f \circ g)^{-1}(9)$ .

- 4 Test whether relation  $R$  defined in  $R$  as  $R = \{(a, b) : a^2 - 4ab + 3b^2 = 0, a, b \in R\}$  is reflexive, summative and transitive
- 5 Let  $A = \{1, 2, 3, 4\}$ ,  $B = \{3, 5, 7, 9\}$  and  $C = \{7, 23, 47, 79\}$  and  $f: A \rightarrow B$ ,  $g: B \rightarrow C$  be defined by  $f(x) = 2x + 1 \forall x \in A$  and  $g(x) = x^2 - 2 \forall x \in B$ . Find  $(g \circ f)^{-1}$  and  $f^{-1} \circ g^{-1}$  at sets of ordered pairs. Is  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ ?
- 6 Let  $A = \{-1, 0, 1, 2\}$   $B = \{-4, -2, 0, 2\}$  and  $f, g: A \rightarrow B$  be functions defined by  $f(x) = x^2 - x$ ,  $x \in A$  and  $g(x) = 2 \left| x - \frac{1}{x} \right| - 1$ ,  $x \in A$ . Find  $g \circ f(x)$  and hence show that  $f = g \circ f$ .
- 7 Consider  $f: R_+ \rightarrow [-9, \infty)$  given by  $f(x) = 5x^2 + 6x - 9$ , where  $R_+$  is the set of all non-negative real numbers. Prove that  $f$  is invertible. Also find the inverse of  $f$ . Hence find  $f^{-1}(2)$  and  $f^{-1}(18)$ .

## ANSWERS

### ONE MARK QUESTIONS

1.  $R_1$  : is universal relation.  
 $R_2$  : is empty relation.  
 $R_3$  : is neither universal nor empty.
2. No,  $R$  is not reflexive.
3.  $(5, 7) \notin R$
4.  $gof = \{(1, 3), (3, 1)\}$
5.  $(fog)(x) = x \forall x \in R$
6.  $f^{-1}(x) = \frac{5x+1}{2}$
7.  $gof(x) = x, fog, (x) = x$
8. 6
9. Yes,  $f$  is one-one  $\because \forall x_1, x_2 \in N \Rightarrow x_1^2 = x_2^2$ .
10.  $A = [1, \infty)$  because  $R_f = [1, \infty)$
11.  $n(B) = 10$
12.  $(fof)(x) = \frac{x-3}{4}$
13. No,  $R$  is not reflexive  $\because (a, a) \notin R \forall a \in N$
14.  $f$  is not one-one function  
 $\because f(3) = f(-1) = 2$   
 $3 \neq -1$  i.e. distinct elements have same images.
15.  $B = [-1, 1]$

16. \_\_\_\_\_
17.  $(1,2) \in R$  and  $(2, 1) \in R$  but  $(1, 1) \notin R$
18. Range =  $\{1, 2, 3\}$
19. Reflexive and symmetric but not transitive.
20.  $\{(1, 1), (2, 2), (3, 3)\}$
21.  $\{(3, 8), (6, 6), (9, 4), \{12, 2\}$
22.  $(f \circ f \circ f)(x) = \frac{x}{\sqrt{3x^2 + 1}}$
23.  $f^{-1}(x) = 7 + (4 - x)^{1/3}$
24. Yes
25. 14
26. 6
27. 0

### TWO MARK QUESTIONS

1. (a) Bijective (one-one, onto)  
 (b) Neither one-one nor onto  
 (c) One-one but not onto  
 (d) Neither one-one nor onto

2.  $g \circ f(x) = 0 \forall x \in R$

$$f \circ g(x) = \begin{cases} 0, & x \geq 0 \\ -4x & x < 0 \end{cases}$$

3.  $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

6. 512, 64

7.  $\pm 4$ , pre image of  $-3$  does not exist.
8.  $f \circ g \left( -\frac{2}{3} \right) = 0, g \circ f \left( -\frac{2}{3} \right) = 1$
9. R is reflexive, not symmetric, not transitive.
10.  $R = \{(8, 6), (8, 4), (8, 2), (6, 4), (6, 2), (4, 2)\}$   
(i) Not reflexive (ii) Not symmetric (iii) Not equivalence relation
11. R is not reflexive, R is not symmetric, R is transitive.

#### 4 MARK QUESTIONS

2. Elements related to 4 are 0, 4, 8.
3.  $f^{-1}(x) = \frac{7x+4}{5x-3}$
6.  $[(3, 4)] = \{(3, 4), (6, 8), (9, 12)\}$
7. Equivalence class  $[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$ .
10.  $f^{-1}(x) = \frac{2x-1}{x-1}$
11.  $a = 1$  and  $b = 0$  or  $a = -1$  and  $b$  can be any real number.
12.  $\{6, 8\}$
13.  $f^{-1}(y) = \begin{cases} y+1, & \text{if } y \text{ is even} \\ y-1, & \text{if } y \text{ is odd} \end{cases}$

#### 6 MARK QUESTIONS

2.  $f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}, f^{-1}(31) = 1$
3.  $(f \circ g)^{-1}(x) = \left( \frac{x+7}{2} \right)^{1/3}, (f \circ g)^{-1}(9) = 2$
4. Reflexive, not symmetric, not transitive
5.  $(g \circ f)^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$   
 $f^{-1} \circ f^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$ , Yes  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .
6.  $(g \circ f)(x) = 2 \left| x^2 - x - \frac{1}{2} \right| - 1$
7.  $f^{-1}(x) = \frac{\sqrt{54+5y}-3}{5}, f^{-1}(2) = 1, f^{-1}(18) = \frac{9}{5}$

## CHAPTER 2

### INVERSE TRIGONOMETRIC FUNCTIONS

#### IMPORTANT POINTS TO REMEMBER

<i>Function</i>	<i>Domain</i>	<i>Range</i> <i>(Principal Value Branch)</i>
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$R$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	$R$	$(0, \pi)$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1}x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- If  $\sin \theta = x$ ,  $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , then  $\theta = \sin^{-1}x$   
 If  $\cos \theta = x$ ,  $\theta \in [0, \pi]$ , then  $\theta = \cos^{-1}x$   
 If  $\tan \theta = x$ ,  $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , then  $\theta = \tan^{-1}x$   
 If  $\cot \theta = x$ ,  $\theta \in (0, \pi)$ , then  $\theta = \cot^{-1}x$   
 If  $\sec \theta = x$ ,  $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ , then  $\theta = \sec^{-1}x$



If  $\operatorname{cosec} \theta = x$ ,  $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ , then find  $\theta = \operatorname{cosec}^{-1} x$

- $\sin^{-1}(\sin x) = x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\cos^{-1}(\cos x) = x \quad \forall x \in [0, \pi]$$

$$\tan^{-1}(\tan x) = x \quad \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cot^{-1}(\cot x) = x \quad \forall x \in (0, \pi)$$

$$\sec^{-1}(\sec x) = x \quad \forall x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

- $\sin(\sin^{-1} x) = x \quad \forall x \in [-1, 1]$

$$\cos(\cos^{-1} x) = x \quad \forall x \in [-1, 1]$$

$$\tan(\tan^{-1} x) = x \quad \forall x \in \mathbb{R}$$

$$\cot(\cot^{-1} x) = x \quad \forall x \in \mathbb{R}$$

$$\sec(\sec^{-1} x) = x \quad \forall x \in \mathbb{R} - (-1, 1)$$

$$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x \quad \forall x \in \mathbb{R} - (-1, 1)$$

- $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right) \quad \forall x \in [-1, 1]$

$$\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right) \quad \forall x > 0$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right), \quad \forall |x| \geq 1$$

- $\sin^{-1}(-x) = -\sin^{-1} x \quad \forall x \in [-1, 1]$

$$\tan^{-1}(-x) = -\tan^{-1} x \quad \forall x \in \mathbb{R}$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x \quad \forall |x| \geq 1$$

- $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad \forall x \in [-1, 1]$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x \quad \forall x \in R$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x \quad \forall |x| \geq 1$$

- $\sin^{-1} X + \cos^{-1} X = \frac{\pi}{2}, X \in [-1, 1]$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad \forall x \in R$$

$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2} \quad \forall |x| \geq 1$$

- $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy < 1 \\ \pi + \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy > 1; x > 0 \\ & y > 0 \\ -\pi + \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy > 1; x < 0 \\ & y < 0 \end{cases}$$

- $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x-y}{1+xy} & \text{if } xy > -1 \\ \pi + \tan^{-1} \frac{x-y}{1+xy} & \text{if } xy < -1; x > 0 \\ & y < 0 \\ -\pi + \tan^{-1} \frac{x-y}{1+xy} & \text{if } xy < -1; x < 0 \\ & y > 0 \end{cases}$$

- $2\tan^{-1} x = \tan^{-1} \left( \frac{2x}{1-x^2} \right), |x| < 1$

$$2\tan^{-1} x = \sin^{-1} \left( \frac{2x}{1+x^2} \right), |x| \leq 1,$$

$$2\tan^{-1} x = \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right), x \geq 0.$$

- $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$$

### ONE MARK QUESTIONS

1. Write the principal value of

(i)  $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(ii)  $\cos^{-1}(\sqrt{3}/2)$

(iii)  $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

(iv)  $\operatorname{cosec}^{-1}(-2)$

(v)  $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(vi)  $\sec^{-1}(-2)$ .

2. What is the value of the following functions (using principal value)

(i)  $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(ii)  $\sin^{-1}\left(-\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii)  $\tan^{-1}(1) - \cot^{-1}(-1)$

(iv)  $\operatorname{cosec}^{-1}(\sqrt{2}) + \sec^{-1}(\sqrt{2})$

(v)  $\tan^{-1}(1) + \cot^{-1}(1) + \sin^{-1}(1)$ .

(vi)  $\sin^{-1}\left(\sin \frac{4\pi}{5}\right)$

(vii)  $\tan^{-1}\left(\tan \frac{5\pi}{6}\right)$

(viii)  $\operatorname{cosec}^{-1}\left(\operatorname{cosec} \frac{3\pi}{4}\right)$

3. If  $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$ , find  $\cot^{-1}x + \cot^{-1}y$ .

4. Find the values of the following

(i)  $\sin \left\{ \frac{\pi}{6} - \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) \right\}$       (ii)  $\tan^{-1} \left( \sin \left( -\frac{\pi}{2} \right) \right)$   
(iii)  $\tan \left( \cos^{-1} \frac{8}{17} \right)$       (iv)  $\sin^{-1} \left( \cos \left( \sin^{-1} \frac{\sqrt{3}}{2} \right) \right)$

5. Evaluate the following

(i)  $\sin (2 \sin^{-1} (0.6))$       (ii)  $\sin (2 \tan^{-1} (0.75))$   
(iii)  $\sin \left( 2 \cos^{-1} \left( -\frac{5}{15} \right) \right)$       (iv)  $\tan \left( \frac{1}{2} \cos^{-1} \left( \frac{\sqrt{5}}{3} \right) \right)$

6. If  $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$ ,  $xy < 1$ , then the value of  $x + y + xy$ .

7. If  $3 \tan^{-1}x + \cot^{-1}x = \pi$ , then find the value of  $x$ .

8. If  $\cos \left( \sin^{-1} \frac{2}{5} + \cos^{-1}x \right) = 0$ , then find the value of  $x$ .

9. If  $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$ , then find the value of  $\cos^{-1}x + \cos^{-1}y$ .

10. If  $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$ , then find the value of  $\alpha(\alpha + \gamma) + \beta(\alpha + \gamma) + \gamma(\alpha + \beta)$ .

11. If  $\tan^{-1}x - \cot^{-1}x = \tan^{-1} \left( \frac{1}{\sqrt{3}} \right)$ , then find the value of  $x$ :

12. Find the value of  $\tan^2 (\sec^{-1}2) + \cot^2 (\operatorname{cosec}^{-1}3)$

13. Evaluate  $\sin \{ \cot^{-1}(\cos (\tan^{-1}1)) \}$

14. If  $a \leq 2 \sin^{-1}x + \cos^{-1}x \leq b$ , then find the value of  $a$  and  $b$ .

15. Solve  $\cos^{-1}(\sin (\cos^{-1}x)) = \frac{\pi}{3}$

16. Write the value of  $\tan \left( 2 \tan^{-1} \frac{1}{5} \right)$

17. Write the value of  $\operatorname{Sec}^{-1} \left( \sec \left( -\frac{8\pi}{5} \right) \right)$

## TWO MARK QUESTIONS

1. Find the value of the following

$$(i) \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) + \cos^{-1} \left( -\frac{1}{2} \right) + \tan^{-1} \left( -\frac{1}{\sqrt{3}} \right)$$

$$(ii) \sin^{-1} \left( \sin \frac{2\pi}{3} \right) + \cos^{-1} \left( \cos \frac{4\pi}{3} \right)$$

$$(iii) \sin \left\{ \frac{\pi}{3} - \sin^{-1} \left( -\frac{1}{2} \right) \right\}$$

$$(iv) \tan^{-1} \left( \tan \frac{7\pi}{6} \right) + \cos^{-1} \left( \cos \frac{7\pi}{6} \right)$$

2. Simplify

$$(i) \tan^{-1} \left( \frac{\sin x}{1 + \cos x} \right)$$

$$(ii) \cot^{-1} \left( \frac{1}{\sqrt{x^2 - 1}} \right), x < -1$$

$$(iii) \cos \left\{ \cos^{-1} \left( \frac{-\sqrt{3}}{2} \right) + \frac{\pi}{6} \right\}$$

$$(iv) \tan \left[ \frac{1}{2} \cos^{-1} \left( \frac{3}{\sqrt{11}} \right) \right]$$

3. Simplify :  $\sin^{-1} \left\{ \frac{\sin x + \cos x}{\sqrt{2}} \right\}, -\frac{\pi}{4} < x < \frac{\pi}{4}$

4. Prove that :  $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$ .

5. Prove that :  $\tan^{-1} \frac{m}{n} - \tan^{-1} \left( \frac{m-n}{m+n} \right) = \frac{\pi}{4}, m, n > 0$

6. Prove that :  $\tan^{-1} \left\{ \frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right\} = \tan^{-1} \left( \frac{a}{b} \right) - x$

7. Evaluate :  $\tan^{-1} \left[ 2 \cos \left( 2 \sin^{-1} \frac{1}{2} \right) \right]$

8. Prove that  $\tan\left(\frac{1}{2} \sin^{-1} \frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$
9. Prove that  $4(\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5}) = \pi$
10. Prove that  $\sin\{\cot^{-1}(\cos(\tan^{-1} x))\} = \sqrt{\frac{x^2+1}{x^2+2}}$
11. Prove that  $\tan^{-1} \frac{2}{3} = \frac{1}{2} \tan^{-1} \frac{12}{5}$

#### 4 MARK QUESTIONS

1. Show that :  $\tan^{-1} \left[ \frac{\sqrt{1+\cos x} + \sqrt{1-\cos x}}{\sqrt{1+\cos x} - \sqrt{1-\cos x}} \right] = \frac{\pi}{4} + \frac{x}{2}; x \in [0, \pi]$

2. Prove that :

$$\tan^{-1} \left( \frac{\cos x}{1-\sin x} \right) - \cot^{-1} \left( \sqrt{\frac{1+\cos x}{1-\cos x}} \right) = \frac{\pi}{4} \quad x \in (0, \pi/2).$$

3. Prove that  $\tan^{-1} \left( \frac{x}{\sqrt{a^2-x^2}} \right) = \sin^{-1} \frac{x}{a} = \cos^{-1} \left( \frac{\sqrt{a^2-x^2}}{a} \right).$

4. prove that:

$$\cot^{-1} \left[ 2 \tan \left( \cos^{-1} \frac{8}{17} \right) \right] + \tan^{-1} \left[ 2 \tan \left( \sin^{-1} \frac{8}{17} \right) \right] = \tan^{-1} \left( \frac{300}{161} \right)$$

5. Prove that:

$$\tan^{-1} \left( \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right) = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2$$

6. Solve:

$$\cot^{-1} 2x + \cot^{-1} 3x = \frac{\pi}{4}$$

7. Prove that:

$$\tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{a}{b}\right)\right] = \frac{2\sqrt{a^2+b^2}}{b}$$

8. Solve for x,  $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{-2x}{1-x^2}\right) = \frac{2\pi}{3}$

9. Prove that:  $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

10. Solve for x,  $\tan(\cos^{-1} x) = \sin(\tan^{-1} 2)$ ;  $x > 0$

11. If  $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$ , then prove that  $\sin y = \tan^2\left(\frac{x}{2}\right)$

12. Prove that:

$$\cot\left\{\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right\} + \cos^{-1}(1 - 2x^2) + \cos^{-1}(2x^2 - 1) = \pi, x > 0$$

13. Prove that:

$$\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right) = 0 \text{ where } a, b, c > 0$$

14. If  $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$ , then

prove that  $a + b + c = abc$

15. If  $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ , prove that  $x^2 + y^2 + z^2 + 2xyz = 1$

[Hint: Let  $\cos^{-1} x = A$ ,  $\cos^{-1} y = B$ ,  $\cos^{-1} z = C$  then  $A + B + C = \pi$  or  $A + B = \pi - C$  take cos on both the sides].

16. If  $\tan^{-1}\left(\frac{1}{1+1.2}\right) + \tan^{-1}\left(\frac{1}{1+2.3}\right) + \dots + \tan^{-1}\left(\frac{1}{1+n.(n+1)}\right) = \tan^{-1} \emptyset$   
then find the value of  $\emptyset$ .

17. If  $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$  then find x.
18. If  $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$ , then find x.
19. Solve the following for x
- $\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3}x) = -\frac{\pi}{2}$
  - $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$
  - $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$
  - $\sin^{-1}\left(\frac{x}{2}\right) + \cos^{-1} x = \frac{\pi}{6}$
20. If  $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \alpha$ , then prove that
- $$9x^2 - 12xy \cos \alpha + 4y^2 = 36 \sin^2 \alpha$$
21. Prove that:  $\tan^{-1} \left[ \frac{3 \sin 2\theta}{5+3 \cos 2\theta} \right] + \tan^{-1} \left[ \frac{1}{4} \tan \theta \right] = \theta$
22. Prove that:  $\cot^{-1} \left[ \cot \left( \sin^{-1} \sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1} \frac{\sqrt{3}}{2} + \cos^{-1} \frac{1}{\sqrt{2}} \right) \right] = \frac{\pi}{2}$
23. Prove that:  $2 \tan^{-1} \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) = \cos^{-1} \left( \frac{a \cos x + b}{a+b \cos x} \right)$
24. Prove that:  $2 \tan^{-1} [\tan \alpha / 2 \tan \beta / 2] = \cos^{-1} \left[ \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta} \right]$
25. Prove that  $\tan \left( \frac{\pi}{4} + \frac{1}{2} \cos^{-1} \frac{a}{b} \right) + \tan \left( \frac{\pi}{4} - \frac{1}{2} \cos^{-1} \frac{a}{b} \right) = \frac{2b}{a}$
26. Prove that  $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$ .
27. Prove that  $2 \tan^{-1} \left[ \tan \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) \tan \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \right] = \tan^{-1} \left( \frac{\cos \alpha \cos \beta}{\sin \alpha + \sin \beta} \right)$



28. Solve for 'x'

$$(i) \tan^{-1} \frac{x-1}{x+1} + \tan^{-1} \frac{2x-1}{2x+1} = \tan^{-1} \frac{23}{36}$$

$$(ii) \tan^{-1} \frac{1}{2x+1} + \tan^{-1} \frac{1}{4x+1} = \tan^{-1} \frac{2}{x^2}$$

### ANSWERS

#### ONE MARK QUESTIONS

1. (i)  $-\frac{\pi}{3}$  (ii)  $\frac{\pi}{6}$  (iii)  $\frac{-\pi}{6}$  (iv)  $\frac{-\pi}{6}$  (v)  $\frac{\pi}{3}$  (vi)  $\frac{2\pi}{3}$

2. (i) 0 (ii)  $\frac{-\pi}{3}$  (iii)  $-\frac{\pi}{2}$  (iv)  $\frac{\pi}{2}$

(v)  $\pi$  (vi)  $\frac{\pi}{5}$  (vii)  $\frac{-\pi}{6}$  (viii)  $\frac{\pi}{4}$

3.  $\pi/5$

4. (i) 1 (ii)  $-\frac{\pi}{4}$  (iii)  $-\frac{15}{8}$  (iv)  $\frac{\pi}{6}$

5. (i) 0.96 (ii) 0.96 (iii)  $-\frac{120}{169}$  (iv)  $\frac{3-\sqrt{5}}{2}$

6. 1

7.  $x = 1$

8.  $x = \frac{2}{5}$

9.  $\frac{\pi}{2}$

10. 6

11.  $x = \sqrt{3}$

12. 11

13.  $\sqrt{\frac{2}{3}}$

14.  $a = 0, b = \pi$

15.  $x = \frac{\sqrt{3}}{2}$

16.  $5/12$

17.  $\frac{2\pi}{5}$

## TWO MARK QUESTIONS

1. (i)  $\frac{\pi}{6}$  (ii)  $\pi$  (iii) 1 (iv)  $\pi$
2. (i)  $\frac{x}{2}$  (ii)  $\pi - \sec^{-1} x$  (iii)  $-1$  (iv)  $\frac{\sqrt{11}-3}{\sqrt{2}}$
3.  $x + \frac{\pi}{4}$  7.  $\frac{\pi}{4}$

## 4 MARK QUESTIONS

6.  $x = \frac{-1}{6}$  8.  $\tan \frac{\pi}{12} = 2 - \sqrt{3}$
10.  $x = \frac{\sqrt{5}}{3}$  14. **Hint:** Let  $\tan^{-1} a = \alpha$   
 $\tan^{-1} b = \beta$

Then given,  $\alpha + \beta + \gamma = \pi$   $\tan^{-1} c = \gamma$

$$\alpha + \beta = \pi - \gamma$$

Take tangent on both sides

$$\tan(\alpha + \beta) = \tan(\pi - \gamma)$$

16.  $\emptyset = \frac{n}{n+2}$  17.  $X = -1$
18.  $x = -\frac{1}{2}$
19. (i)  $x = -\frac{1}{12}$  (ii)  $x = 0, \frac{1}{2}$
- (iii)  $x = 13$  (iv)  $x = 1$
28. (i)  $x = \frac{4}{3}, x \in \left(-1, \frac{-1}{2}\right) \cup (0, \infty)$
- (ii)  $x = 3, x \in \left(-\infty, \frac{-3}{4}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup (0, \infty)$

## CHAPTER: 3 and 4

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### MATRICES AND DETERMINANTS

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#### IMPORTANT POINTS TO REMEMBER

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**Matrix:** It is an ordered rectangular arrangement of numbers (or functions). The numbers (or functions) are called the **elements** of the matrix. Horizontal line of elements is **row** of matrix. Vertical line of elements is **column** of matrix.

Numbers written in the horizontal line form a row of the matrix.  
Number written in the vertical line form a column of the matrix.

**Order of Matrix** with ' $m$ ' rows and ' $n$ ' columns is  $m \times n$  (read as  $m$  by  $n$ ).

#### Types of Matrices

- A **row matrix** has only one row (order:  $1 \times n$ )
- A **column matrix** has only one column (order:  $m \times 1$ )
- A **square matrix** has number of rows equal to number of columns (order:  $m \times m$  or  $n \times n$ .)
- A **diagonal matrix** is a square matrix with all non-diagonal elements equal to zero and diagonal elements not all zeroes.
- A **scalar matrix** is a diagonal matrix in which all diagonal elements are equal.
- An **identity matrix** is a scalar matrix in which each diagonal element is 1 (unity).
- A **zero matrix** or **null matrix** is the matrix having all elements zero.

- **Equal matrices:** two matrices  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$  are equal if
  - (a) Both have same order
  - (b)  $a_{ij} = b_{ij} \forall i \text{ and } j$

### Operations on matrices

- Two matrices can be added or subtracted, if both have same order.
- If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$ , then
 
$$A \pm B = [a_{ij} \pm b_{ij}]_{m \times n}$$
- $\lambda A = [\lambda a_{ij}]_{m \times n}$  where  $\lambda$  is a scalar
- Two matrices A and B can be multiplied if number of columns in A is equal to number of rows in B.

$$\text{If } A = [a_{ij}]_{m \times n} \text{ and } [b_{jk}]_{n \times p}$$

$$\text{Then } AB = [c_{ik}]_{m \times p} \text{ where } c_{ik} = \sum_{j=1}^n a_{ij} b_{jk}$$

### Properties

- If A, B and C are matrices of same order, then
  - (i)  $A+B = B+A$
  - (ii)  $(A+B)+C = A+(B+C)$
  - (iii)  $A+O = O+A=A$
  - (iv)  $A+(-A) = O$

- If A, B and C are matrices and  $\lambda$  is any scalar, then

- $AB \neq BA$
- $(AB)C = A(BC)$
- $A(B+C) = AB+AC$
- $AB=O$  need not necessarily imply  $A=O$  or  $B=O$
- $\lambda (AB) = (\lambda A)B = A(\lambda B)$

**Transpose of a Matrix:** Let A be any matrix. Interchange rows and columns of A. The new matrix so obtained is transpose of A denoted by  $A' / A^T$ .

[order of A =  $m \times n \Rightarrow$  order of  $A' = n \times m$ ]

Properties of transpose matrices A and B are:

- $(A')' = A$
- $(kA)' = kA'$  ( $k = \text{constant}$ )
- $(A + B)' = A' + B'$
- $(AB)' = B'A'$

### Symmetric Matrix and Skew-Symmetric matrix

- A square matrix  $A = [a_{ij}]$  is symmetric if  $A' = A$  i.e.  $a_{ij} = a_{ji} \forall i$  and  $j$
- A square matrix  $A = [a_{ij}]$  is skew-symmetric if  $A' = -A$  i.e.  $a_{ij} = -a_{ji} \forall i$  and  $j$   
(All diagonal elements are zero in skew-symmetric matrix)

**Determinant:** to every square matrix  $A = [a_{ij}]$  of order  $n \times n$ , we can associate a number (real or complex). This is called determinant of A ( $\det A$  or  $|A|$ ).

### Properties of Determinants

- I)  $|AB| = |A| |B|$
- II)  $|A'| = |A|$
- III) If we interchange any two rows (or columns), sign of  $|A|$  changes.
- IV) Value of  $|A|$  is zero, if any two rows or columns in  $A$  are identical (or proportional).
- V) 
$$\begin{vmatrix} a+b & x \\ c+d & y \end{vmatrix} = \begin{vmatrix} a & x \\ c & y \end{vmatrix} + \begin{vmatrix} b & x \\ d & y \end{vmatrix}$$
- VI)  $R_i \rightarrow R_i \pm aR_j$  or  $C_i \rightarrow C_i \pm bC_j$  does not alter the value of  $|A|$ .
- VII)  $|k A|_{n \times n} = k^n |A|_{n \times n}$  ( $k$  = scalar)
- VIII)  $K |A|$  means multiplying only one row (or column) by  $k$ .
- IX) Area of triangle with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  is:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  are collinear if area of triangle is zero

### Minors and Cofactors

- Minor ( $M_{ij}$ ) of  $a_{ij}$  in Matrix  $A$  is the determinant of order  $(n-1)$  obtained by leaving  $i^{th}$  row and  $j^{th}$  column of  $A$ .
- Cofactor of  $a_{ij}$ ,  $A_{ij} = (-1)^{i+j} M_{ij}$

### Adjoint of a Square Matrix

adj  $A$  = transpose of the square matrix  $A$  whose elements have been replaced by their cofactors.

**Properties of adj A:** For any square matrix A of order n:

$$(i) \quad A(\text{adj } A) = (\text{adj } A) A = |A| I$$

$$(ii) \quad |\text{adj } A| = |A|^{n-1}$$

$$(iii) \quad \text{adj}(AB) = (\text{adj } B) (\text{adj } A).$$

$$(iv) \quad |k \text{ adj } A| = k^n |A|^{n-1}.$$

**Singular Matrix:** A square matrix A is singular if  $|A| = 0$ .

**Inverse of a Matrix:** An inverse of a square matrix exists if and only if it is non-singular.

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

**Properties of Inverse matrix**

$$(i) \quad AA^{-1} = A^{-1}A = I$$

$$(ii) \quad (A^{-1})^{-1} = A$$

$$(iii) \quad (AB)^{-1} = B^{-1}A^{-1}$$

$$(iv) \quad (A')^{-1} = (A^{-1})'$$

$$(v) \quad |A'| = \frac{1}{|A|}, |A| \neq 0$$

**Solution of system of equations using matrices:**

If  $AX = B$  is a matrix equation, then

$$AX=B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B \text{ gives the solution.}$$

**Criterion of consistency of system of linear equations**

$$(i) \quad \text{If } |A| \neq 0, \text{ system is consistent and has a unique solution.}$$

- (ii) If  $|A| = 0$  and  $(adj A) B \neq 0$ , then the system  $AX=B$  is inconsistent and has no solution.
- (iii) If  $|A| = 0$  and  $(adj A) B = 0$  then system is consistent and has infinitely many solutions.

### ONE MARK QUESTIONS

1. If  $[1 \ x \ 1] \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$ , then What is the value of  $x$ ?
2. For what value of  $\lambda$ , the matrix  $A$  is a singular matrix where

$$A = \begin{bmatrix} 1 & 3 & \lambda + 2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$$

3. Find the value of  $A^2$ , if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

4. If  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$  and  $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ , then find the value of  $\alpha$  and  $\beta$ .
5. If  $A$  is a square matrix such that  $A^2 = I$ , then write the value of  $(A - I)^3 + (A + I)^3 - 7A$  in simplest form.
6. Write the value of  $\Delta$ , if

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

7. If  $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$ , find the value of  $x+y$ .



8. If  $A$  is a  $3 \times 3$  matrix,  $|A| \neq 0$  and  $|3A| = K|A|$ , then write the value of  $K$ .

9. If  $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$  is a symmetric matrix, then write the value of  $x$ .

10. Matrix  $A = \begin{bmatrix} 0 & 2a & -2 \\ 3 & 1 & 3 \\ 3b & 3 & -1 \end{bmatrix}$  is given to be symmetric, find the value of  $a$  and  $b$ .

11. For any  $2 \times 2$  matrix  $A$ , if  $A(\text{adjoint } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ , then find  $|A|$ .

12. Find  $X$ , if  $A + X = I$ , where

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 3 & 4 & 7 \\ 5 & 1 & 6 \end{bmatrix}$$

13. If  $U = [2 \quad -3 \quad 4]$ ,  $V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ,  $X = [0 \quad 2 \quad 3]$  and  $Y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ , then find  $UV + XY$ .

14. If  $\begin{bmatrix} 2 & -3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -9 \\ 16 & 15 \end{bmatrix}$

write the equation after applying elementary column transformation  
 $C_2 \rightarrow C_2 + 2C_1$

15. If  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ , then find the value of  $A^3$ .

16. Find the value of  $a_{23} + a_{32}$  in the matrix

$$A = [a_{ij}]_{3 \times 3} \text{ where } a_{ij} = \begin{cases} |2i - j| & \text{if } i > j \\ -i + 2j + 3 & \text{if } i < j \end{cases}$$

17. If  $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ , then find  $|A^2|$ .

18. For what value of  $x$ , is the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & x & -3 \\ 2 & 3 & 0 \end{bmatrix} \text{ a skew-symmetric matrix}$$

19. If  $A = \begin{bmatrix} \sin 15^\circ & \cos 15^\circ \\ -\sin 75^\circ & \cos 75^\circ \end{bmatrix}$ , then evaluate  $|A|$ .
20. If  $A$  is a square matrix, expressed as  $A = X + Y$  where  $X$  is symmetric and  $Y$  is skew-symmetric, then write the values of  $X$  and  $Y$ .
21. Write a matrix of order  $3 \times 3$  which is both symmetric and skew-symmetric matrix.
22. What positive value of  $x$  makes the following pair of determinants equal?

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \quad \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

23.  $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ , find the value of  $5A_{31} + 3A_{32} + 8A_{33}$ .

24. If  $A = \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix}$ , find  $|A (adjA)|$

25. Find the minimum value of.  $2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 & 1 & 1 + \cos\theta \end{vmatrix}$

26. If  $A$  and  $B$  are square matrices of order 3 and  $|A| = 5$  and  $|B| = 3$ , then find the value of  $|3AB|$ .

27. Evaluate  $\begin{vmatrix} 3 + 2i & -6i \\ 2i & 3 - 2i \end{vmatrix}$ ,  $i = \sqrt{-1}$

28. Without expanding, find the value of  $\begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$

29. Using determinants, find the equation of line passing through  $(0, 3)$  and  $(1, 1)$ .

30. If A be any square matrix of order  $3 \times 3$  and  $|A| = 5$ , then find the value of  $|adj(adjA)|$
31. What is the number of all possible matrices of order  $2 \times 3$  with each entry 0,1 or 2.
32. Given a square matrix A of order  $3 \times 3$  such that  $|A|=12$ , find the value of  $|A adj A|$
33. If  $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$  find  $|(A^{-1})^{-1}|$
34. If  $A = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$  find  $|AB|$
35. Find  $|A (adjoint A)|$  and  $|adjoint A|$ , if  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

## TWO MARK QUESTIONS

1. Construct a matrix of order  $2 \times 3$ , whose elements are given by
- (a)  $a_{ij} = \frac{(i-2j)^2}{2}$                       (b)  $a_{ij} = \frac{|-2i+j|}{3}$
2. If A  $(x_1, y_1)$ , B  $(x_2, y_2)$  and C  $(x_3, y_3)$  are vertices of an equilateral triangle with each side equal to a units, then prove that

$$\begin{bmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{bmatrix}^2 = 3a^4$$

3. Prove that the diagonal elements of a skew-symmetric matrix are all zero.

4. If  $2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

Find the value of  $x - y$

5. If A and B are skew symmetric matrices of the same order prove that  $AB + BA$  is symmetric matrix.

6. Without expanding prove that  $\begin{bmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{bmatrix} = 0$

7. Let  $A = \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix}$  Prove that  $A+A'$  is symmetric matrix.

8. If  $A = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$  and  $B = [1 \ 2 \ 3]$ , Verify  $(AB)' = B'A'$

9. If  $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 5 & -4 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$  and  $C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Find  $AB-AC$ .

10. If  $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$  Find the determinant of  $A^2-2A$

11. Without expanding, evaluate  $\begin{bmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{bmatrix}$

12. If  $D_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$  and  $D_2 = \begin{vmatrix} m & -b & y \\ -l & a & -x \\ n & -c & z \end{vmatrix}$  evaluate  $D_1+D_2$ .

13. If A is a skew symmetric matrix of odd order, then prove that  $|A| = 0$

14. Write the minors and co-factors of each element of the first column of the matrix A

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

15. Find x and y, if  $\begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$

16. If  $A = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ , find matrix 'C', such that  $2A+3C=5B$
17. If  $A = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$  find x such that  $A^2=B$ .
18. Construct a matrix of order  $3 \times 2$ , whose elements  $a_{ij}$  given by
- $$a_{ij} = \begin{cases} 2i-3j, & i \geq j \\ 3i+j, & i < j \end{cases}$$

#### 4 MARK QUESTIONS

1. If  $\begin{vmatrix} a & y & z \\ x & b & z \\ x & y & c \end{vmatrix} = 0$ , then prove that  $\frac{a}{a-x} + \frac{b}{b-y} + \frac{c}{c-z} = 2$
2. If  $a \neq b \neq c$ , find the value of x which satisfies the equation
- $$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$$
3. Using properties of determinants, show that
- $$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 0$$
4. Find the value of  $\begin{vmatrix} \sqrt{13} + \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{26} & 5 & \sqrt{10} \\ 3 + \sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$
5. If  $A = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$ , show that  $A^2 - 12A - I = 0$ . Hence find  $A^{-1}$ .

6. Find the matrix  $X$  so that  $X \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 2 & 0 \end{bmatrix}$
7. If  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ , verify that  $A^2 - 4A - 5I = 0$ .
8. Using elementary transformations find the inverse of the matrix
- $$A = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$$
9. If  $A = \begin{bmatrix} x & -2 \\ 3 & 7 \end{bmatrix}$  and  $A^{-1} = \begin{bmatrix} \frac{7}{34} & \frac{1}{17} \\ -\frac{3}{34} & \frac{2}{17} \end{bmatrix}$ , then find the value of  $x$ .
10. If  $A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$ , find  $B$ , such that  $4A^{-1} + B = A^2$
11. If  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ ,  $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$  and  $B = A^{-1}$ , then find the value of  $\alpha$ .
12. Find the value of  $X$ , such that  $A^2 - 5A + 4I + X = 0$ , if  $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$
13. If  $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ , find  $(A')^{-1}$
14. The monthly incomes of Mohan and Sohan are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves ₹ 15000/- per month, find their monthly incomes and expenditures using matrices.

15. If  $A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$ , then verify that  $(AB)' = B'A'$

16. If  $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $x^2 = -1$

Then show that  $(A + B)^2 = A^2 + B^2$

17. Prove that  $aI + bA + cA^2 = A^3$ , if  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$

18. If  $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$ , then find  $A^3$ .

19. If  $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$  and  $(A + B)^2 = A^2 + B^2 + 2AB$ , find  $a$  and  $b$ .

20. If  $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ , then find the value of  $a$ ,  $b$  and  $c$ . Such that  $A^T A = I$

21. If  $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ , then prove that  $A^n = \begin{bmatrix} a^n & b(\frac{a^n-1}{a-1}) \\ 0 & 1 \end{bmatrix}$ , for all  $n \in \mathbb{N}$ .

22. Find the value of  $k$ , if:  $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

23. If  $x, y$  and  $z \in \mathbb{R}$ , and

$$\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16, \text{ then find value of } x.$$

24. Find the value of 'k' if  $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = k \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Using properties of determinants, prove the following (Q. 68 to 31)

25.  $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix} = 0$

26.  $\begin{vmatrix} 1 & a^2 + bc & a^3 \\ 1 & b^2 + ac & b^3 \\ 1 & c^2 + ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2 + b^2 + c^2)$

27.  $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

28.  $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

29.  $\begin{vmatrix} a^2 & bc & c^2 + ac \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

30.  $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3abc - a^3 - b^3 - c^3)$



$$31. \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

$$32. \text{ Given } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}. \text{ Find the product } AB \text{ and also find } (AB)^{-1}$$

33. Using properties of determinants, solve for x:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

$$34. \text{ If } \begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix} = 0 \text{ and } a \neq b \neq c \text{ then find the value of } x.$$

35. Express the following matrix as the sum of symmetric and skew-symmetric matrices and verify your result.

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$36. \text{ If } x = -4 \text{ is a root of } \Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0, \text{ then find the other two roots.}$$

37. Using properties of determinants. Find the value of 'x'

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4-x & 4+x & 4+x \end{vmatrix} = 0.$$

38. Using proportion determinants prove that

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(2x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix} = 6x^2(1-x^2)$$

39. If  $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$  using properties of determinant

Find the value of  $f(2x) - f(x)$

40. If  $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$  show that  $A^2 - 5A + 4I = 0$ .

Hence find  $A^{-1}$

41. If  $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$  show that  $A^2 = A^{-1}$

## 6 MARK QUESTIONS

1. Prove that  $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$  is divisible by  $(x + y + z)$  and hence find the quotient.

2. Using elementary transformations, find the inverse of the matrix

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

3. Using matrix method, solve the system of linear equations

$$x - 2y = 10, 2x - y - z = 8 \text{ and } -2y + z = 7$$

4. Find  $A^{-1}$  if  $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  and show that  $A^{-1} = \frac{A^2 - 3I}{2}$

5. Find the matrix  $x$  for which  $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

6. Let  $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$  and  $f(x) = x^2 - 4x + 7$ , then show that  $f(A) = 0$ , using this result find  $A^5$ .

7. If  $a + b + c = 0$  and  $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ , then show that either

$$x = 0 \quad \text{or} \quad x = \pm \sqrt{\frac{3}{2}(a^2 + b^2 + c^2)}$$

8. If  $A + B + C = \pi$ , then find the value of

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$$

9. If  $\Delta = \begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ 2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$  prove that  $\Delta$  is negative.

10. Using properties of determinants prove that:

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$$

11. Prove that:  $\begin{vmatrix} a & a+c & a-b \\ b-c & b & b+a \\ c+b & c-a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

12. If  $a, b, c$  are  $p^{\text{th}}, q^{\text{th}}$  and  $r^{\text{th}}$  terms respectively of a G.P. Prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

13. Prove that  $(x-2)(x-1)$  is factor of  $\begin{vmatrix} 1 & 1 & x \\ \beta+1 & \beta+1 & \beta+x \\ 3 & x+1 & x+2 \end{vmatrix}$  and hence find the quotient.

14. Prove that:

$$\begin{vmatrix} -a(b^2 + c^2 - a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2 + a^2 - b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2 + b^2 - c^2) \end{vmatrix} = abc(a^2 + b^2 + c^2)^3$$

15. Determine the product  $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

and use it to solve the system of equations:

$$x - y + z = 4, \quad x - 2y - 2z = 9, \quad 2x + y + 3z = 1$$

16. If  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ , find  $A^{-1}$  and use it to solve the system of linear equations:  $x + 2y + z = 4$ ,  $-x + y + z =$ ,  $x - 3y + z = 2$

17. Solve given system of equations by matrix method:

$$\frac{2}{a} + \frac{3}{b} + \frac{4}{c} = -3, \quad \frac{5}{a} + \frac{4}{b} - \frac{6}{c} = 4, \quad \frac{3}{a} - \frac{2}{b} - \frac{2}{c} = 6$$

18. To raise money for an orphanage, students of three schools A, B and C organized an exhibition in their locality, where they sold paper bags, scrap books and pastel sheets made by them using recycled paper, at the rate of ₹ 20, ₹ 15 and ₹ 5 per unit respectively. School A sold 25 paper bags, 12 scrap books and 34 pastel sheets. School B sold 22 paper bags, 15 scrap books and 28 pastel sheets. While school C sold 26 paper bags, 18 scrap books and 36 pastel sheets. Using matrices, find the total amount raised by each school.

19. Two cricket teams honored their players for three values, excellent batting, to the point bowling and unparalleled fielding by giving ₹ x, ₹ y and ₹ z per player respectively. The first team paid respectively 2, 2 and 1 players for the above values with a total prize money of 11 lakhs, while the second team paid respectively 1, 2 and 2 players for these values with a total prize money of ₹ 9 lakhs. If the total award money for one person each for these values amount to ₹ 6 lakhs, then express the above situation as a matrix equation and find award money per person for each value.

20. It  $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$  find  $A^{-1}$  using elementary transformation

Hence solve the system of linear equations.

$$x - 2y = 10$$

$$2x - y - z = 8$$

$$-2y + z = 7$$

### ANSWERS

### ONE MARK QUESTIONS

- |                                      |  |
|--------------------------------------|--|
| 1. $\frac{1}{2}$                     | 9. $X = 5$   |
| 2. $\lambda = 4$                     | 10. $a = \frac{3}{2}, b = \frac{-2}{3}$  |
| 3. $A^2 = I_3$                       | 11. $ A  = 10$   |
| 4. $\alpha = a^2 + b^2, \beta = 2ab$ | 12. $X = \begin{bmatrix} 0 & -4 & 1 \\ -3 & -3 & -7 \\ -5 & -1 & -5 \end{bmatrix}$                 |
| 5. A                                 | 13. [20]   |
| 6. 0                                 | 14. $\begin{bmatrix} 2 & -3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} =$ |
| 7. 3                                 | $\begin{bmatrix} -4 & -17 \\ 16 & 47 \end{bmatrix}$  |
| 8. $K = 27$                          |  |

15.  $A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

16. 11

17. 0

18.  $x = 0$

19.  $|A| = 1$

20.  $x = \frac{1}{2}(A + A'), y = \frac{1}{2}(A - A')$

21.  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

22.  $x = \pm 4$

23. 0

24. 9

25. -1

26. 405

27. 1

28. 0

29.  $3 - 2x$

30. 625

31. 729

32. 1728

33. 11

34. -11

35.  $a^9, a^6$

## 2 MARK QUESTIONS

1 (a)  $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{bmatrix}$

(b)  $\begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

4.  $x - y = -7$

9.  $\begin{vmatrix} 1 & -2 & -8 \\ -2 & 0 & -21 \\ 0 & 1 & 16 \end{vmatrix}$

10. 25

11. 0

12.  $D_1 + D_2 = 2$

14.  $M_{11} = -12, M_{21} = -16, M_{31} = -4$   
 $C_{11} = -12, C_{21} = 16, C_{31} = -4$

15.  $x=2, y=2$

16.  $C = \begin{vmatrix} 12 & 4/3 \\ 4 & -14/3 \\ 25/3 & 28/3 \end{vmatrix}$

17. No value of  $x$ , for which  $A^2=B$ .

18.  $A = \begin{vmatrix} -1 & 5 \\ 1 & -2 \\ 3 & 0 \end{vmatrix}$

#### 4 MARKS QUESTION

2.  $x = 0$

4.  $-5\sqrt{3}(5-\sqrt{6})$

5.  $A^{-1} = \begin{bmatrix} -7 & 3 \\ 12 & -5 \end{bmatrix}$

6.  $\begin{bmatrix} 5 & 0 \\ -6 & 4 \\ 7 & 7 \end{bmatrix}$

8.  $\frac{1}{10} \begin{bmatrix} 7 & -1 \\ -4 & 2 \end{bmatrix}$

9.  $x = 4$

10.  $B = \begin{bmatrix} 2 & -15 \\ 0 & -3 \end{bmatrix}$

11.  $\alpha = 5$

12.  $X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$

13.  $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

14. Incomes: Rs 90,000/- and  
Rs 1,20,000/-  
Expenditures: Rs 75,000/-  
and Rs 10,5000/-

18.  $\begin{bmatrix} \cos 8\theta & \sin 8\theta \\ -\sin 8\theta & \cos 8\theta \end{bmatrix}$

19.  $a = -1, b = -2$

20.  $a = \pm \frac{1}{\sqrt{2}}; \quad b = \pm \frac{1}{\sqrt{6}};$

$c = \pm \frac{1}{\sqrt{3}}$

22.  $K = 2$

23.  $x = 2$

24.  $K = (a + b)(b + c)(c + a)$

32.  $AB = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$$(AB)^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$$

33.  $x = 4$

34.  $x = \frac{-abc}{ab+bc+ca}$

35.  $A = \frac{1}{2} \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$

36.  $x = 1, 3$

37.  $x = 0, -12$

39.  $ax(2a + 3x)$

40.  $A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

### 6 MARK QUESTIONS

1.  $(x + y + z)(xy + yz + zx - x^2 - y^2 - z^2)^2$

2.  $A^{-1} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}$



3.  $x = 0; y = -5; z = -3$

4.  $A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

5.  $x = \begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix}$

6.  $\begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$

8. 0

13.  $\beta$

15. *Product* = 81

$x = 3, y = -2, z = -1$

16.  $A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \quad x = \frac{9}{5}, y = \frac{2}{5}, z = \frac{7}{5}$

17.  $a = 1, b = -1, c = -2$

18. School A = ₹ 850

19. Excellent batting: 3 lakhs

School B = ₹ 805

point bowling: 2 lakhs

School C = ₹ 970

fielding: 1lakh

20.  $A^{-1} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$   
 $x=0, y=-5, z=-3.$

## CHAPTER 5

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### CONTINUITY AND DIFFERENTIABILITY

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#### POINTS TO REMEMBER

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- A function  $f(x)$  is said to be continuous at  $x = c$  iff  $\lim_{x \rightarrow c} f(x) = f(c)$   
*i.e.*,  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$
- $f(x)$  is continuous in  $(a, b)$  iff it is continuous at  $x = c \forall c \in (a, b)$ .
- $f(x)$  is continuous in  $[a, b]$  iff
  - (i)  $f(x)$  is continuous in  $(a, b)$
  - (ii)  $\lim_{x \rightarrow a^+} f(x) = f(a)$
  - (iii)  $\lim_{x \rightarrow b^-} f(x) = f(b)$
- Modulus functions is Continuous on  $\mathbb{R}$
- Trigonometric functions are continuous in their respective domains.
- Exponential function is continuous on  $\mathbb{R}$
- Every polynomial function is continuous on  $\mathbb{R}$ .
- Greatest integer function is continuous on all non-integral real numbers
- If  $f(x)$  and  $g(x)$  are two continuous functions at  $x = a$  and if  $c \in \mathbb{R}$  then
  - (i)  $f(x) \pm g(x)$  are also continuous functions at  $x = a$ .
  - (ii)  $g(x) \cdot f(x), f(x) + c, cf(x), |f(x)|$  are also continuous at  $x = a$ .
  - (iii)  $\frac{f(x)}{g(x)}$  is continuous at  $x = a$  provided  $g(a) \neq 0$ .
- A function  $f(x)$  is derivable or differentiable at  $x = c$  in its domain iff

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}, \text{ and is finite}$$

The value of above limit is denoted by  $f'(c)$  and is called the derivative of  $f(x)$  at  $x = c$ .

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

- $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$  (Product Rule)
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$  (Quotient Rule)
- If  $y = f(u)$  and  $u = g(t)$  then  $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u)g'(t)$  (Chain Rule)
- If  $y = f(u)$ ,  $x = g(u)$  then,  

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{f'(u)}{g'(u)}$$

- **Rolle's theorem:** If  $f(x)$  is continuous in  $[a, b]$  derivable in  $(a, b)$  and  $f(a) = f(b)$  then there exists at least one real number  $c \in (a, b)$  such that  $f'(c) = 0$ .
- **Mean Value Theorem :** If  $f(x)$  is continuous in  $[a, b]$  and derivable in  $(a, b)$  then there exists at least one real number  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Every differentiable function is continuous but its converse is not true.

## 1 MARK QUESTIONS

1. Let  $f(x) = \sin x \cos x$ . Write down the set of points of discontinuity of  $f(x)$ .
2. Given  $f(x) = \frac{1}{x+2}$ , write down the set of points of discontinuity of  $f(f(x))$ .

3. Write the set of points of continuity of  
 $f(x) = |x - 1| + |x + 1|$
4. Write the number of points of discontinuity of  $f(x) = [x]$  in  $[3, 7]$ .
5. If  $y = e^{\log(x^5)}$ , find  $\frac{dy}{dx}$ .
6. If  $f(x) = x^2 g(x)$  and  $g(1) = 6$ ,  $g'(x) = 3$ , find the value of  $f'(1)$ .
7. If  $y = a \sin t$ ,  $x = a \cos t$  then find  $\frac{dy}{dx}$
8. Find value of  $f(0)$ , so that  $\frac{-e^x + 2^x}{x}$  may be continuous at  $x=0$ .
9. Find the values of  $x$  for which  $f(x) = \frac{x^2 + 7}{x^3 + 3x^2 - x - 3}$  is discontinuous.
10. If  $y = \tan^{-1}x + \cot^{-1}x + \sec^{-1}x$ ,  $\operatorname{cosec}^{-1}x$  then  $dy/dx$  is equal to  
 (a)  $\pi$  (b) 0 (c) 1 (d)  $\frac{x^2 - 1}{x^2 + 1}$
11. If  $y = \log_e e^{\sin x^2}$ , find  $\frac{dy}{dx}$
12.  $y = \log_a x + \log_x a + \log_x x + \log_a a$ , then  $\frac{dy}{dx} = ?$   
 (a)  $\frac{1}{x} + x \log a$  (b)  $\frac{1}{x \log a} + x \log a$  (c)  $\frac{\log a}{x} + \frac{x}{\log a}$  (d) None of these
13. If  $y = 5^x \cdot x^5$ , then find  $\frac{dy}{dx}$
14. What is derivative of  $\sin^{-1}(2x\sqrt{1-x^2})$  w.r.t.  $\sin^{-1}(3x - 4x^3)$ ?
15. If  $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}}$ , then  $(2y - 1) \frac{dy}{dx}$  is equal to  
 (a)  $\sin x$  (b)  $-\sin x$  (c)  $\cos x$  (d)  $-\cos x$

## 2 MARK QUESTIONS

1. Differentiate  $\sin(x^2)$  w. r. t.  $e^{\sin x}$
2.  $y = x^y$  then find  $\frac{dy}{dx}$

3. If  $y = x^x + x^3 + 3^x + 3^3$ , find  $\frac{dy}{dx}$
4. If  $y = 2\sin^{-1}(\cos x) + 5 \operatorname{cosec}^{-1}(\sec x)$ . Find  $\frac{dy}{dx}$
5. If  $y = e^{[\log(x+1) - \log x]}$  find  $\frac{dy}{dx}$
6. Differentiate  $\sin^{-1}[x\sqrt{x}]$  w.r.t.  $x$ .
7. Find the derivative of  $|x^2+2|$  w.r.t.  $x$
8. Find the domain of the continuity of  $f(x) = \sin^{-1}x - [x]$
9. Find the derivative of  $\cos(\sin x^2)$  w.r.t.  $x$  at  $x = \sqrt{\frac{\pi}{2}}$
10. If  $y = e^{3\log x + 2x}$ , Prove that  $\frac{dy}{dx} = x^2(2x+3)e^{2x}$ .
11. Differentiate  $\sin^2(\theta^2+1)$  w.r.t.  $\theta^2$
12. Find  $\frac{dy}{dx}$  if  $y = \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)$
13. If  $x^2 + y^2 = 1$  verify that  $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$
14. Find  $\frac{dy}{dx}$  when  $y = 10^{x^{10^x}}$
15. Find  $y = x^x$  find  $\frac{d^2y}{dx^2}$
16. Find  $\frac{dy}{dx}$  if  $y = \cos^{-1}(\sin x)$
17. If  $f(x) = x + 7$ , and  $g(x) = x - 7$ ,  $x \in \mathbb{R}$ , then find  $\frac{d}{dx} (f \circ g)(x)$ .
18. Differentiate  $\log(7 \log x)$  w.r.t.  $x$
19. If  $y = f(x^2)$  and  $f'(x) = \sin x^2$ . Find  $\frac{dy}{dx}$
20. Find  $\frac{dy}{dx}$  if  $y = \sqrt{\sin^{-1}\sqrt{x}}$

## 4 MARK QUESTIONS

1. Examine the continuity of the following functions at the indicated points.

$$(I) \quad f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{at } x = 0$$

$$(II) \quad f(x) = \begin{cases} x - [x] & x \neq 1 \\ 0 & x = 1 \end{cases} \quad \text{at } x = 1$$

$$(III) \quad f(x) = \begin{cases} \frac{e^x - 1}{e^x + 1} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \quad \text{at } x = 0$$

$$(IV) \quad f(x) = \begin{cases} \frac{x - \cos(\sin^{-1}x)}{1 - \tan(\sin^{-1}x)} & x \neq \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & x = \frac{1}{\sqrt{2}} \end{cases} \quad \text{at } x = \frac{1}{\sqrt{2}}$$

2. For what values of constant K, the following functions are continuous at the indicated points.

$$(I) \quad f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} & x < 0 \\ \frac{2x+1}{x-1} & x > 0 \end{cases} \quad \text{at } x = 0$$

$$(II) \quad f(x) = \begin{cases} \frac{e^x - 1}{\log(1+2x)} & x \neq 0 \\ K & x = 0 \end{cases} \quad \text{at } x = 0$$

$$(III) \quad f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & x < 0 \\ K & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & x > 0 \end{cases} \quad \text{at } x = 0$$

3. For what values a and b

$$f(x) = \begin{cases} \frac{x+2}{|x+2|} + a & \text{if } x < -2 \\ a + b & \text{if } x = -2 \\ \frac{x+2}{|x+2|} + 2b & \text{if } x > -2 \end{cases}$$

Is continuous at  $x = -2$

4. Find the values of a, b and c for which the function

$$f(x) = \begin{cases} \frac{\sin[(a+1)x] + \sin x}{x} & x < 0 \\ c & x = 0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & x > 0 \end{cases}$$

Is continuous at  $x = 0$

5.  $f(x) = \begin{cases} [x] + [-x] & x \neq 0 \\ \lambda & x = 0 \end{cases}$

Find the value of  $\lambda$ ,  $f$  is continuous at  $x = 0$  ?

6. Let  $f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x} ; & x < \frac{\pi}{2} \\ a & x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} ; & x > \frac{\pi}{2} \end{cases}$

If  $f(x)$  is continuous at  $x = \frac{\pi}{2}$ , find  $a$  and  $b$ .

7. If  $f(x) = \begin{cases} x^3 + 3x + a & x \leq 1 \\ bx + 2 & x > 1 \end{cases}$

Is everywhere differentiable, find the value of  $a$  and  $b$ .

8. Find the relationship between  $a$  and  $b$  so that the function defined by

$$f(x) = \begin{cases} ax+1, & x \leq 3 \\ bx+3, & x > 3 \end{cases} \text{ is continuous at } x = 3.$$

9. Differentiate  $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$  w.r.t  $\cos^{-1}(2x\sqrt{1-x^2})$  where  $x \neq 0$ .

10. If  $y = x^{x^x}$ , then find  $\frac{dy}{dx}$ .

11. Differentiate  $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$  w.r.t.  $x$ .

12. If  $(x + y)^{m+n} = x^m \cdot y^n$  then prove that  $\frac{dy}{dx} = \frac{y}{x}$
13. If  $(x - y) \cdot e^{\frac{x}{x-y}} = a$ , prove that  $y \left( \frac{dy}{dx} \right) + x = 2y$
14. If  $x = \tan \left( \frac{1}{a} \log y \right)$  then show that
- $$(1 + x^2) \frac{d^2 y}{dx^2} + (2x - a) \frac{dy}{dx} = 0$$
15. If  $y = x \log \left( \frac{x}{a+bx} \right)$  prove that  $x^3 \frac{d^2 y}{dx^2} = \left( x \frac{dy}{dx} - y \right)^2$ .
16. Differentiate  $\sin^{-1} \left[ \frac{2^{x+1} \cdot 3^x}{1+(36)^x} \right]$  w.r.t  $x$ .
17. If  $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$ , prove that
- $$\frac{dy}{dx} = \frac{x^2}{y^2} \sqrt{\frac{1-y^6}{1-x^6}}, \text{ Where } -1 < x < 1 \text{ and } -1 < y < 1 \text{ [HINT: put } x^3 = \sin A \text{ and } y^3 = \sin B]$$
18. If  $f(x) = \sqrt{x^2 + 1}$ ,  $g(x) = \frac{x+1}{x^2+1}$  and  $h(x) = 2x - 3$  find  $f'[h'(g'(x))]$ .
19. If  $x = \sec \theta - \cos \theta$  and  $y = \sec^n \theta - \cos^n \theta$ , then prove that  $\frac{dy}{dx} = n \sqrt{\frac{y^2+4}{x^2+4}}$
20. If  $x^y + y^x + x^x = m^n$ , then find the value of  $\frac{dy}{dx}$ .
21. If  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$  then find  $\frac{d^2 y}{dx^2}$  at  $x = \frac{\pi}{6}$
22. If  $y = \tan^{-1} \left[ \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right]$  where  $0 < x < \frac{\pi}{2}$  find  $\frac{dy}{dx}$



23. If  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , then show that  $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$ .

24. Verify Rolle's theorem for the function

$$f(x) = e^x \sin 2x \left[0, \frac{\pi}{2}\right]$$

25. Verify mean value theorem for the function

$$f(x) = \sqrt{x^2 - 4} \quad [2, 4]$$

26. If the Rolle's theorem holds for the function

$$f(x) = x^3 + bx^2 + ax + 5 \text{ on } [1, 3] \text{ with } c = \left(2 + \frac{1}{\sqrt{3}}\right)$$

Find the value of  $a$  and  $b$ .

27. If  $y = [x + \sqrt{x^2 + 1}]^m$ , show that  $(x^2 + 1)y_2 + xy_1 - m^2y = 0$ .

28. Differentiate  $\sin^{-1} \left[ \frac{3x+4\sqrt{1-x^2}}{5} \right]$  w.r.t.  $x$ .

29. If  $x^y = e^{x-y}$ , prove that  $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

30. If  $f: [-5, 5] \rightarrow R$  is a differentiable function and  $f'(x)$  does not vanish anywhere, then prove that  $f(-5) \neq f(5)$ .

31. If  $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$  then prove that  $(x^2 - 1)y_2 + xy_1 = m^2y$ .

## ANSWERS

### 1 MARK QUESTIONS

1.  $\{ \}$
2.  $\{-2, -\frac{5}{2}\}$
3.  $R$
4. Points of discontinuity of  $f(x)$  are 4,5,6,7  
Note- At  $x = 3, f(x) = [x]$  is continuous because  $\lim_{x \rightarrow 3^+} f(x) = 3 = f(3)$
5.  $5x^4$
6. 15
7.  $-\cot t$
8.  $-1 + \log 2$
9.  $x = -1, 1, -3$
10. (b)
11.  $2x \cos(x^2)$
12. (d)
13.  $5^x (x^5 \log 5 + 5x^4)$
14.  $2/3$
15. (c)

### 2 MARK QUESTIONS

1.  $\frac{2x \cos(x^2)}{\cos x e^{\sin x}}$
2.  $\frac{y^2}{x[1 - y \log x]}$
3.  $x^x [1 + \log x] + 3x^2 + 3^x \log 3$
4.  $-\frac{7}{x^2}$
5.  $-\frac{1}{x^2}$
6.  $\frac{3}{2} \sqrt{\frac{x}{1-x^3}}$
7.  $\frac{2x(x^2+2)}{|x^2+2|}$
8.  $(-1, 0) \cup (0, 1)$
9. 0
11.  $\sin(2\theta^2 + 2), \theta \neq 0$
12. 0
14.  $10^{x^{10^x}} x^{10^x} 10^x \log 10 \left[ \frac{1}{x} + \log 10 \log x \right]$
15.  $x^x \left[ (1 + \log x)^2 + \frac{1}{x} \right]$
16. -1
17.  $\frac{1}{x \log x}$
18.  $\frac{1}{x \log x}$
19.  $2x \sin x^4$
20.  $\frac{1}{4\sqrt{x}\sqrt{1-x}\sqrt{\sin^{-1}\sqrt{x}}}$ , where  $0 < x < 1$

### 4 MARK QUESTIONS

1. (I) Continuous  
(II) Discontinuous  
(III) Not Continuous at  $x = 0$   
(IV) Continuous
2. (I)  $K = -1$   
(II)  $K = 1/2$   
(III)  $K = 8$
3.  $a = 0, b = -1$
4.  $a = \frac{-3}{2}, b = R - \{0\}, c = \frac{1}{2}$
5.  $\lambda = -1$
6.  $a = \frac{1}{2}, b = 4$
7.  $a = 3, b = 5$
8.  $3a - 3b = 2$
9.  $-\frac{1}{2}$
10.  $x^x x^{x^x} \left\{ (1 + \log x) \log x + \frac{1}{x} \right\}$
11.  $(x \cos x)^x [1 - x \tan x + (\log x \cos x)] + (x \sin x)^{1/x} \left[ \frac{1+x \cot x - \log(x \sin)^x}{x^2} \right]$
16.  $\left[ \frac{2^{x+1} 3^x}{1+(36)^x} \right] \log 6$  [Hint:  $\tan \theta = 6^x$ ]
18.  $\frac{2}{\sqrt{5}}$
20.  $\frac{dy}{dx} = \frac{x^x(1+\log x) + yx^{y-1} - y^x \log y}{x^y \log x + xy^{x-1}}$
21.  $\frac{32}{27a}$
22.  $-1/2$
26.  $a = 11, b = -6$
28.  $\frac{1}{\sqrt{1-x^2}}$

## CHAPTER 6

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### APPLICATION OF DERIVATIVES

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#### IMPORTANT POINTS TO REMEMBER

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- **Rate of change:** Let  $y = f(x)$  be a function then the rate of change of  $y$  with respect to  $x$  is given by  $\frac{dy}{dx} = f'(x)$  where a quantity  $y$  varies with another quantity  $x$ .

$$\left\{ \frac{dy}{dx} \right\}_{x=x_1} \text{ or } f'(x_1) \text{ represents the rate of change of } y \text{ w.r.t. } x \text{ at } x = x_1.$$

- **Increasing and Decreasing Function**

Let  $f$  be a real-valued function and let  $I$  be any interval in the domain of  $f$ . Then  $f$  is said to be

- a) Strictly increasing on  $I$ , if for all  $x_1, x_2 \in I$ , we have

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

- b) Increasing on  $I$ , if for all  $x_1, x_2 \in I$ , we have

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

- c) Strictly decreasing in  $I$ , if for all  $x_1, x_2 \in I$ , we have

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

- d) Decreasing on  $I$ , if for all  $x_1, x_2 \in I$ , we have

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

- **Derivative Test:** Let  $f$  be a continuous function on  $[a, b]$  and differentiable on  $(a, b)$ . Then

- a)  $f$  is strictly increasing on  $[a, b]$  if  $f'(x) > 0$  for each  $x \in (a, b)$ .
- b)  $f$  is increasing on  $[a, b]$  if  $f'(x) \geq 0$  for each  $x \in (a, b)$ .
- c)  $f$  is strictly decreasing on  $[a, b]$  if  $f'(x) < 0$  for each  $x \in (a, b)$ .
- d)  $f$  is decreasing on  $[a, b]$  if  $f'(x) \leq 0$  for each  $x \in (a, b)$ .
- e)  $f$  is constant function on  $[a, b]$  if  $f'(x) = 0$  for each  $x \in (a, b)$ .

- **Tangents and Normals**

- a) Equation of the tangent to the curve  $y = f(x)$  at  $(x_1, y_1)$  is

$$y - y_1 = \left[ \frac{dy}{dx} \right]_{(x_1, y_1)} (x - x_1)$$

- b) Equation of the normal to the curve  $y = f(x)$  at  $(x_1, y_1)$  is

$$y - y_1 = \frac{-1}{\left[ \frac{dy}{dx} \right]_{(x_1, y_1)}} (x - x_1)$$

- **Maxima and Minima**

- a) Let  $f$  be a function and  $c$  be a point in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist are called critical points.

- b) **First Derivative Test:** Let  $f$  be a function defined on an open interval

I. Let  $f$  be continuous at a critical point  $c$  in I. Then

- i.  $f'(x)$  changes sign from positive to negative as  $x$  increases through  $c$ , then  $c$  is called the point of the local maxima.
- ii.  $f'(x)$  changes sign from negative to positive as  $x$  increases through  $c$ , then  $c$  is a point of *local minima*.

- iii.  $f'(x)$  does not change sign as  $x$  increases through  $c$ , then  $c$  is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion*.

**c) Second Derivative Test :** Let  $f$  be a function defined on an interval  $I$  and let  $c \in I$ . Let  $f$  be twice differentiable at  $c$ . Then

- i.  $x = c$  is a point of local maxima if  $f'(c) = 0$  and  $f''(c) < 0$ . The value  $f(c)$  is local maximum value of  $f$ .
- ii.  $x = c$  is a point of local minima if  $f'(c) = 0$  and  $f''(c) > 0$ . The value  $f(c)$  is local minimum value of  $f$ .
- iii. The test fails if  $f'(c) = 0$  and  $f''(c) = 0$ .

## 1 MARK QUESTIONS

1. Find the angle  $\theta$ , where  $0 < \theta < \frac{\pi}{2}$ , which increases twice as fast as its sine.
2. Find the slope of the normal to the curve  $x = a \cos^3 \theta$  and  $y = a \sin^3 \theta$  at  $\theta = \frac{\pi}{4}$
3. A balloon which always remains spherical has a variable radius. Find the rate at which its volume is increasing with respect to its radius when the radius is 7cm.
4. Write the interval for which the function  $f(x) = \cos x, 0 \leq x \leq 2\pi$  is decreasing

5. For what values of  $x$  is the rate of increasing of  $x^3 - 5x^2 + 5x + 8$  is twice the rate of increase of  $x$ ?
6. Find the point on the curve  $y = x^2 - 2x + 3$  where the tangent is parallel to x-axis.
7. Write the maximum value of  $f(x) = \frac{\log x}{x}$ , if it exists.
8. Find the least value of  $f(x) = ax + \frac{b}{x}$ , where  $a > 0$ ,  $b > 0$  and  $x > 0$ .
9. Find the interval in which the function  $f(x) = x - e^x + \tan(\frac{2\pi}{7})$  increases.
10. Find the value of  $a$  for which the function  $f(x) = x^2 - 2ax + 6$ ,  $x > 0$  is strictly increasing.
11. Find the minimum value of  $\sin x + \cos x$ .
12. Which of the following function is decreasing on  $(0, \frac{\pi}{2})$ ?  
 (a)  $\sin 2x$  (b)  $\cos 3x$   
 (c)  $\tan x$  (d)  $\cos 2x$
13. Find the absolute maximum of  $x^{40} - x^{20}$  on the interval  $(0, 1)$ .
14. The angle between  $y^2 = x$  and  $x^2 = y$  at the origin is  
 (a)  $2 \tan^{-1} \frac{3}{4}$  (b)  $\tan^{-1} \frac{4}{3}$   
 (c)  $\frac{\pi}{2}$  (d)  $\frac{\pi}{4}$
15. Find the local minimum value of  $f'(x)$  if  $f(x) = 3 + |x|$ ,  $x \in \mathbb{R}$ .
16. The distance covered by a particle in  $t$  sec. is given by  $x = 3 + 8t - 4t^2$ . What will be its velocity after 1 second.
17. If the rate of change of volume of a sphere is equal to the rate of change of its radius, then find  $r$ .

## 2 MARK QUESTIONS

1. Find the co-ordinates of the point on the curve  $y^2 = 3 - 4x$ , where tangent is parallel to the line  $2x + y - 2 = 0$
2. The sum of the two numbers is 8, what will be the maximum value of the sum of their reciprocals.
3. Find the maximum value of  $f(x) = 2x^3 - 24x + 107$  in the interval  $[1, 3]$
4. If the rate of change of Area of a circle is equal to the rate of change its diameter. Find the radius of the circle.
5. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when side is 10 cm.
6. If there is an error of  $a\%$  in measuring the edge of cube, then what is the percentage error in its surface?
7. If an error of  $k\%$  is made in measuring the radius of a sphere, then what is the percentage error in its volume?
8. Find the point on the curve  $y^2 = x$  where tangent makes  $45^\circ$  angle with  $x$ -axis.
9. Find the slope of tangent to the curve  $x = 3t^2 + 1$ ,  $y = t^3 - 1$  at  $x = 1$
10. If the curves  $y = 2e^x$  and  $y = ae^{-x}$  intersect orthogonally, then find  $a$ .
11. Find the point on the curve  $y^2 = 8x$  for which the abscissa and ordinate change at the same rate.
12. Prove that the function  $f(x) = \tan x - 4x$  is strictly decreasing on  $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ .
13. Find the point on the curve  $y = x^2$ , where the slope of the tangent is equal to the  $x$  co-ordinate of the point.
14. Use differentials to approximate the cube root of 66.
15. Find the maximum and minimum values of the function  $f(x) = \sin(\sin x)$
16. Find the local maxima and minima of the function  $f(x) = 2x^3 - 21x^2 + 36x - 20$ .
17. If  $y = a \log x + bx^2 + x$  has its extreme values at  $x = -1$  and  $x = 2$ , then find  $a$  and  $b$ .
18. Find the equation of the tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at the point  $(x_0, y_0)$ .
19. If the radius of the circle increases from 5 into 5.1 cm, then find the increase in area.
20. Find the equation of the normal to the curve  $y = 2x^3 + 3 \sin x$  at  $x = 0$ .



## 4 MARK QUESTIONS

1. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of  $900 \text{ cm}^3$  of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm. Why is child labour not good for society?
2. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of  $\frac{3}{2}$  c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.
3. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of an edge of the cube.
4. A kite is moving horizontally at a height of 151.5 meters. If the speed of the kite is 10m/sec, how fast is the string being let out when the kite is 250 m away from the boy who is flying the kite ? The height of the boy is 1.5 m.
5. A swimming pool is to be drained for cleaning. If  $L$  represents the number of litres of water in the pool  $t$  seconds after the pool has been plugged off to drain and  $L = 200(10 - t)^2$ . How fast is the water running out at the end of 5 sec. and what is the average rate at which the water flows out during the first 5 seconds?
6. A man 2m tall, walk at a uniform speed of 6km/h away from a lamp post 6m high. Find the rate at which the length of his shadow increases.
7. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lower most. Its semi- vertical angle is  $\tan^{-1}(0.5)$ . water is poured into it at a constant rate of  $5\text{m}^3/\text{h}$ . Find the

rate at which the level of the water is rising at the instant, when the depth of Water in the tank is 4m.

8. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.
9. A conical vessel whose height is 10 meters and the radius of whose base is half that of the height is being filled with a liquid at a uniform rate of  $1.5m^3/min$ . find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.
10. Let  $x$  and  $y$  be the sides of two squares such that  $y = x - x^2$ . Find the rate of change of area of the second square w.r.t. the area of the first square.
11. The length of a rectangle is increasing at the rate of 3.5 cm/sec. and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.
12. If the areas of a circle increases at a uniform rate, then prove that the perimeter varies inversely as the radius.
13. Show that  $f(x) = x^3 - 6x^2 + 18x + 5$  is an increasing function for all  $x \in R$ . Find its value when the rate of increase of  $f(x)$  is least.  
[Hint: Rate of increase is least when  $f'(x)$  is least.]
14. Determine whether the following function is increasing or decreasing in the given interval:  $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$ ,  $\frac{3\pi}{8} \leq x \leq \frac{5\pi}{8}$ .
15. Determine for which values of  $x$ , the function  $y = x^4 - \frac{4x^3}{3}$  is increasing and for which it is decreasing.

16. Find the interval of increasing and decreasing of the function

$$f(x) = \frac{\log x}{x}$$

17. Find the interval of increasing and decreasing of the function  $f(x) = \sin x - \cos x$ ,  $0 < x < 2\pi$ .

18. Show that  $f(x) = x^2 e^{-x}$ ,  $0 \leq x \leq 2$  is increasing in the indicated interval.

19. Prove that the function  $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$  is an increasing function of  $\theta$  in  $\left[0, \frac{\pi}{2}\right]$ .

20. Find the intervals in which the following function is decreasing.

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

21. Find the interval in which the function  $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}$ ,  $x > 0$  is strictly decreasing.

22. Show that the function  $f(x) = \tan^{-1}(\sin x + \cos x)$ , is strictly increasing the interval  $\left(0, \frac{\pi}{4}\right)$ .

23. Find the interval in which the function  $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$  is increasing or decreasing.

24. Find the interval in which the function given by

$$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$$

- (1) strictly increasing  
(2) strictly decreasing

25. Find the equation of the tangent to the curve  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at the point  $(\sqrt{2}a, b)$ .
26. Find the equation of the tangent to the curve  $y = x^2 - 2x + 7$  which is
- (1) Parallel to the line  $2x - y + 9 = 0$
  - (2) Perpendicular to the line  $5y - 15x = 13$
27. Find the co-ordinates of the point on the curve  $\sqrt{x} + \sqrt{y} = 4$  at which tangent is equally inclined to the axes.
28. Find a point on the parabola  $f(x) = (x - 3)^2$  where the tangent is parallel to the chord joining the points  $(3,0)$  and  $(4,1)$ .
29. Find the equation of the normal to the curve  $y = e^{2x} + x^2$  at  $x = 0$ , also find the distance from origin to the line.
30. Show that the line  $\frac{x}{a} + \frac{y}{b} = 1$  touches the curve  $y = be^{-x/a}$  at the point, where the curve intersects the axis of  $y$ .
31. At what point on the circle  $x^2 + y^2 - 2x - 4y + 1 = 0$  the tangent is parallel to
- (1) X – axis
  - (2) Y – axis
32. Show that the equation of the normal at any point ' $\theta$ ' on the curve  $x = 3 \cos \theta - \cos^3 \theta$ ,  $y = 3 \sin \theta - \sin^3 \theta$  is  $4(y \cos^3 \theta - x \sin^3 \theta) = 3 \sin 4 \theta$ .
33. Show that the curves  $xy = a^2$  and  $x^2 + y^2 = 2a^2$  touch each other.
34. For the curve  $y = 5x - 2x^3$ , if  $x$  increases at the rate of 2 Units/sec. then how fast is the slope of the curve changing when  $x=3$ ?

35. Find the condition for the curve  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  and  $xy = c^2$  to intersect orthogonally.
36. Show that the curves  $y = a^x$  and  $y = b^x$ ,  $a > b > 0$  intersect at an angle of  $\tan^{-1} \left( \left| \frac{\log \left| \frac{a}{b} \right|}{1 + \log \log b} \right| \right)$
37. Find the equation of the normal to the curve  $ay^2 = x^3$  at the point  $(am^2, am^3)$ .
38. Find the equation of the normal at a point on the curve  $x^2 = 4y$ , which passes through the point  $(1, 2)$ . Also find the equation of the corresponding tangent.
39. Find the point on the curve  $9y^2 = x^3$  where the normal to the curve makes equal intercepts with the axes.
40. Show that the tangents to the curve  $y = 2x^3 - 3$  at the point where  $x = 2$  and  $x = -2$  are parallel.

Use differentials to find the approximate value of (Ques.57 to 62)

- |                     |                     |
|---------------------|---------------------|
| 41. $(66)^{1/3}$    | 42. $\sqrt{401}$    |
| 43. $\sqrt{0.037}$  | 44. $\sqrt{25.3}$   |
| 45. $(3.968)^{3/2}$ | 46. $(26.57)^{1/3}$ |
47. Find the value of  $\log_{10}(10.1)$  given that  $\log_{10}e = 0.4343$ .
48. If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

49. If the side of a cube be increased by 0.1%, find the corresponding increase in the volume of the cube.
50. Find the approximate value of  $f(2.01)$  where  $f(x) = x^3 - 4x + 7$ .
51. Find the approximate value of  $\frac{1}{\sqrt{25.1}}$ , using differentials.
52. The radius of a sphere shrinks from 10 cm. to 9.8 cm. Find the approximately decrease in its volume.
53. Find the maximum and minimum values of  $f(x) = \sin x + \frac{1}{2} \cos 2x$  in  $\left[0, \frac{\pi}{2}\right]$ .
54. Find the absolute maximum value and absolute minimum value of the following question  $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$  in  $[-2, 2.5]$
55. Find the maximum and minimum values of  $f(x) = x^{50} - x^{20}$  in the interval  $[0, 1]$
56. Find the absolute maximum and absolute minimum value of  $f(x) = (x - 2)\sqrt{x - 1}$  in  $[1, 9]$
57. Find the difference between the greatest and least values of the function  $f(x) = \sin 2x - x$  on  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ .

### 6 MARK QUESTIONS

1. Prove that the least perimeter of an isosceles triangle in which a circle of radius  $r$  can be inscribed is  $6\sqrt{3}r$ .
2. If the sum of length of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is  $\frac{\pi}{3}$ .

3. Show that semi-vertical angle of a cone of maximum volume and given slant height is  $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ .
4. The sum of the surface areas of cuboids with sides  $x$ ,  $2x$  and  $\frac{x}{3}$  and a sphere is given to be constant. Prove that the sum of their volumes is minimum if  $x = 3$  radius of the sphere. Also find the minimum value of the sum of their volumes.
5. Show that the volume of the largest cone that can be inscribed in a sphere of radius  $R$  is  $\frac{8}{27}$  of the volume of the sphere.
6. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to  $\frac{2}{3}$  of the diameter of the sphere.
7. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.
8. Show that the volume of the greatest cylinder which can be inscribed in a cone of height  $H$  and semi-vertical angle  $\alpha$  is  $\frac{4}{27}\pi h^3 \tan^2 \alpha$ . Also show that height of the cylinder is  $\frac{h}{3}$ .
9. Find the point on the curve  $y^2 = 4x$  which is nearest to the point  $(2,1)$ .
10. Find the shortest distance between the line  $y - x = 1$  and the curve  $x = y^2$ .
11. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum?
12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius  $r$  is  $\frac{2r}{\sqrt{3}}$ .
13. Find the area of greatest rectangle that can be inscribed in an ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

## Answers

### 1 MARK QUESTIONS

1.  $\frac{\pi}{3}$
2. 1
3.  $196\pi \frac{cm^3}{cm}$
4.  $[0, \pi]$
5.  $3, \frac{1}{3}$
6. (1,2)
7.  $\frac{1}{e}$
8.  $2\sqrt{ab}$
9.  $(-\infty, 0)$
10.  $a \leq 0$
11.  $-\sqrt{2}$
12. (d)
13. 0
14. (c)
15. -1
16. 0 unit
17.  $\frac{1}{2\sqrt{\pi}} \text{ unit}$

### 2 MARK QUESTIONS

1.  $\left(\frac{1}{2}, 1\right)$
2.  $\frac{1}{2}$

3. 89
4.  $\frac{1}{\pi}$  units
5.  $10\sqrt{3} \text{ cm}^2 / s$
6.  $2a\%$
7.  $3k\%$
8.  $\left(\frac{1}{4}, \frac{1}{2}\right)$
9. 0
10.  $\frac{1}{2}$
11. (2, 4)
13. (0, 0)
14.  $\approx 4.042$
15.  $\sin 1, -\sin 1$
16. Local maxima at  $x = 1$   
Local minima at  $x = 6$
17.  $a = 2, b = -\frac{1}{2}$
18.  $\frac{xx_0}{a^2} - \frac{yy_0}{b^2}$
19.  $\pi \text{ cm}^2$
20.  $x + 3y = 0$

### 4 MARK QUESTIONS

1.  $\frac{1}{\pi} \text{ cm} / s$
2.  $\frac{3}{8\pi} \text{ cm} / \text{min}$
4. 8 m/sec.
5. 3000 L/s
6. 3 km/h



7.  $\frac{35}{88} \text{ m/h}$
9.  $\frac{6}{49\pi} \text{ m/min.}$
10.  $1 - 3x + 2x^2$
11.  $8 \text{ cm}^2/\text{sec}$
13. 25
14. Increasing
15. Increasing for all  $x \geq 1$   
Decreasing for all  $x \leq 1$
16. Increasing on  $(0, e)$   
Decreasing on  $[e, \infty)$
17. Increasing on  
 $\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$   
Decreasing on  $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$
20.  $(-\infty, 1] \cup [2, 3]$
21.  $[1, \infty]$
23. increasing on  $[0, \infty)$   
Decreasing  $(-\infty, 0]$
24. Strictly increasing  
 $[-2, 1] \cup [3, \infty)$   
Strictly decreasing  
 $(-\infty, -2] \cup [1, 3]$
25.  $\sqrt{2}bx - ay - ab = 0$
26. (1)  $y - 2x - 3 = 0$   
(2)  $36y + 12x - 227 = 0$
27.  $(4, 4)$
28.  $\left(\frac{7}{2}, \frac{1}{4}\right)$
29.  $2y + x - 2 = 0, \frac{2}{\sqrt{5}}$
31. (1)  $(1, 0)$  and  $(1, 4)$   
(2)  $(3, 2)$  and  $(-1, 2)$
34. decrease 72 units/sec.
35.  $a^2 = b^2$
37.  $2x + 3my - 3a m^4 - 2am^2 = 0$
38.  $x + y = 3, \quad y = x - 1$
39.  $\left(4, \pm \frac{8}{3}\right)$
41. 4.042
42. 20.025
43. 0.1924
44. 5.03

45. 7.904  
46. 2.984  
47. 1.004343

48.  $\pi \text{ cm}^2$

49. 0.3%

50. 7.08

51. 0.198

52.  $80\pi \text{ cm}^3$

53. max. value  $= \frac{3}{4}$ , min value  $= \frac{1}{2}$

54. ab. Max.  $= \frac{157}{8}$ , ab. Min.  $= \frac{-7}{4}$

55. max.value=0,

min.value  $= \frac{-3}{5} \left[ \frac{2}{5} \right]^{2/3}$

56. ab. Max = 14 at  $x = 9$

ab. Min.  $= \frac{-3}{4^{4/3}}$  at  $x = \frac{5}{4}$

57.  $\pi$

## 6 MARK QUESTIONS

4.  $18r^3 + \frac{4}{3}\pi r^3$

9. (1, 2)

10.  $\frac{3\sqrt{2}}{8}$

11.  $\frac{144}{\pi+4}m, \frac{36\pi}{\pi+4}m$

13. 2ab sq. Units.

## CHAPTER 7

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## INTEGRALS

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### POINTS TO REMEMBER

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- Integration or anti derivative is the reverse process of Differentiation.
- Let  $\frac{d}{dx} F(x) = f(x)$  then we write  $\int f(x) dx = F(x) + c$ .
- These integrals are called indefinite integrals and  $c$  is called constant of integration.
- From geometrical point of view, an indefinite integral is the collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along  $y$ -axis.

### STANDARD FORMULAE

1.  $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log_e|x| + c & n = -1 \end{cases}$
2.  $\int (ax + b)^n dx = \begin{cases} \frac{(ax+b)^{n+1}}{(n+1)a} + c & n \neq -1 \\ \frac{1}{a} \log|ax + b| + c & n = -1 \end{cases}$
3.  $\int \sin x dx = -\cos x + c$ .
4.  $\int \cos x dx = \sin x + c$
5.  $\int \tan x \cdot dx = -\log|\cos x| + c = \log|\sec x| + c$ .
6.  $\int \cot x dx = \log|\sin x| + c$ .
7.  $\int \sec^2 x dx = \tan x + c$

$$8. \quad \int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

$$9. \quad \int \sec x \tan x \, dx = \sec x + c$$

$$10. \quad \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$11. \quad \int \sec x \, dx = \log|\sec x + \tan x| + c$$

$$= \log \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right| + c$$

$$12. \quad \int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$$

$$= \log \left| \tan \frac{x}{2} \right| + c$$

$$13. \quad \int e^x \, dx = e^x + c$$

$$14. \quad \int a^x \, dx = \frac{a^x}{\log a} + c$$

$$15. \quad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c, |x| < 1$$

$$= -\cos^{-1} x + c$$

$$16. \quad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c$$

$$= -\cot^{-1} x + c$$

$$17. \quad \int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1} x + c, |x| > 1$$

$$= -\operatorname{cosec}^{-1} x + c$$

$$18. \quad \int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$19. \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$20. \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$21. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c$$

$$22. \int \frac{1}{\sqrt{a^2 + x^2}} dx = \log |x + \sqrt{a^2 + x^2}| + c$$

$$23. \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}| + c$$

$$24. \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

$$25. \int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2 + x^2}| + c$$

$$26. \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c$$

### RULES OF INTEGRATION

$$1. \int [(f_1(x) \pm f_2(x) \pm \dots \pm f_x(x))] dx = \int f_1(x) dx \pm \int f_2(x) dx \pm \dots \pm \int f_x(x) dx$$

$$2. \int k \cdot f(x) dx = k \int f(x) dx.$$

$$3. \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$$

### INTEGRATION BY SUBSTITUTION

$$1. \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$$

$$2. \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$3. \int \frac{f'(x)}{[f(x)]^n} dx = \frac{(f(x))^{-n+1}}{-n+1} + c$$

## INTEGRATION BY PARTS

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int \left[ f'(x) \int g(x) dx \right]$$

## DEFINITE INTEGRALS

$$\int_a^b f(x) dx = F(b) - F(a), \text{ where } F(x) = \int f(x) dx$$

## DEFINITE INTEGRAL AS A LIMIT OF SUMS.

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+n-1h)]$$

$$\text{Where } h = \frac{b-a}{n} \text{ or } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} [h \sum_{r=1}^n f(a+rh)]$$

## PROPERTIES OF DEFINITE INTEGRAL

$$1. \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$2. \int_a^b f(x) dx = \int_a^b f(t) dt.$$

$$3. \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

$$4. (i) \int_a^b f(x) dx = \int_a^b f(a+b-x) dx.$$

$$(ii) \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$5. \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \quad \text{if } f(x) \text{ is even function}$$

$$6. \int_{-a}^a f(x) dx = 0 \text{ if } f(x) \text{ is an odd function}$$

$$7. \int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

### ONE MARK QUESTIONS

Evaluate the following integrals:

$$1. \int (\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}) dx$$

$$2. \int_{-1}^1 e^{|x|} dx$$

$$3. \int \frac{dx}{1-\sin^2 x}$$

$$4. \int_{-1}^1 x^{99} \cos^4 x dx$$

$$5. \int \frac{1}{x \log x \log(\log x)} dx$$

$$6. \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \log \left( \frac{1+x}{1-x} \right) dx$$

$$7. \int (e^{a \log x} + e^{x \log a}) dx$$

$$8. \int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} dx$$

9.  $\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$
10.  $\int \sqrt{10 - 4x + x^2} \, dx$
11.  $\int_{-1}^1 x^3 |x| \, dx$
12.  $\int \frac{1}{\sin^2 x \cos^2 x} \, dx$
13.  $\int_{-2}^2 \frac{dx}{1+|x-1|}$
14.  $\int e^{-\log x} \, dx$
15.  $\int \frac{e^x}{a^x} \, dx$
16.  $\int \frac{x}{\sqrt{x+1}} dx$
17.  $\int \frac{x}{(x+1)^2} \, dx$
18.  $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$
19.  $\int \cos^2 \alpha \, dx$
20.  $\int \frac{1}{x \cos \alpha + 1} \, dx$
21.  $\int \sec x \log(\sec x + \tan x) \, dx$
22.  $\int \frac{1}{\cos \alpha + x \sin \alpha} \, dx$
23.  $\int \frac{\sec^2(\log x)}{x} \, dx$
24.  $\int \frac{e^x}{\sqrt{4+e^{2x}}} \, dx$
25.  $\int \frac{1}{x(2+3 \log x)} \, dx$
26.  $\int \frac{1-\sin x}{x+\cos x} dx$



27.  $\int \frac{1-\cos x}{\sin x} dx$
28.  $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x} dx$
29.  $\int \frac{(x+1)}{x} (x + \log x) dx$
30.  $\int_0^\pi |\cos x| dx$
31.  $\int_0^2 [x] dx$  where  $[x]$  is greatest integers function.
32.  $\int \frac{1}{\sqrt{9-4x^2}} dx$
33.  $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$
34.  $\int_{-2}^1 \frac{|x|}{x} dx$
35.  $\int_{-1}^1 x |x| dx$
36.  $\int x \sqrt{x+2} dx$
37.  $\int_a^b f(x) dx + \int_b^a f(x) dx$
38.  $\int \frac{\sin x}{\sin 2x} dx$
39.  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x| dx$
40.  $\int \frac{1}{\sec x + \tan x} dx$
41.  $\int \frac{\sin^2 x}{1+\cos x} dx$
42.  $\int \frac{1-\tan x}{1+\tan x} dx$

## TWO MARK QUESTIONS

Evaluate :

- |  |   |
|--|---|
| 1. $\int e^{[\log(x+1) - \log x]} dx$  | 11. $\int x \log 2x \, dx$                                    |
| 2. $\int \frac{1}{\sqrt{x+1} + \sqrt{x+2}} dx$                                     | 12. $\int_0^{\pi/4} \sqrt{1 + \sin 2x} dx$                    |
| 3. $\int \sin x \sin 2x dx$  | 13. $\int_0^{\pi/2} e^x (\sin x - \cos x) dx$                 |
| 4. $\int \left[ \frac{x}{a} + \frac{a}{x} + x^a + a^x \right] dx$                  | 14. $\int_4^9 \frac{\sqrt{x}}{(30 - x^{3/2})} dx$             |
| 5. $\int_0^{\pi/2} \text{Log} \left( \frac{5 + 3 \cos x}{5 + 3 \sin x} \right) dx$ | 15. $\int_0^1 \frac{dx}{e^x + e^{-x}}$                        |
| 6. $\int \frac{a^x + b^x}{c^x} dx$   | 16. $\int \frac{\log  \sin x }{\tan x} dx$                    |
| 7. $\int \left( \sqrt{ax} - \frac{1}{\sqrt{ax}} \right)^2 dx$                      | 17. $\int \frac{\sin^4 x + \cos^4 x}{\sin^3 x + \cos^3 x} dx$ |
| 8. $\int e^x 2^x dx$   | 18. $\int \sqrt{\tan x} (1 + \tan^2 x) dx$                    |
| 9. $\int 2^{2^{2^x}} 2^{2^x} 2^x dx$   | 19. $\int \frac{\sin 2x}{(a + b \cos x)^2} dx$                |
| 10. $\int \frac{\sin(2 \tan^{-1} x)}{1 + x^2} dx$                                  | 20. $\int \frac{x^2 - x + 2}{x^2 + 1} dx$                     |

## FOUR MARK QUESTIONS

Evaluate :

1. (I)  $\int \frac{x \operatorname{cosec}(\tan^{-1} x^2)}{1 + x^4} dx$
- (II)  $\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx$
- (III)  $\int \frac{1}{\sin(x-a) \sin(x-b)} dx$

$$(IV) \quad \int \frac{\cos(x+a)}{\cos(x-a)} dx$$

$$(V) \quad \int \cos 2x \cos 4x \cos 6x dx$$

$$(VI) \quad \int \tan 2x \tan 3x \tan 5x dx$$

$$(VII) \quad \int \sin^2 x \cos^4 x dx$$

$$(VIII) \quad \int \cot^3 x \operatorname{cosec}^4 x dx$$

$$(IX) \quad \int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx \quad [\text{Hint: Put } a^2 \sin^2 x + b^2 \cos^2 x = t \text{ or } t^2]$$

$$(X) \quad \int \frac{1}{\sqrt{\cos^3 x \cos(x+a)}} dx$$

$$(XI) \quad \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

$$(XII) \quad \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Evaluate :

$$2. \quad (i) \quad \int \frac{x}{x^4 + x^2 + 1} dx$$

$$(ii) \quad \int \frac{1}{x[6(\log x)^2 + 7 \log x + 2]} dx$$

$$(iii) \quad \int \frac{1}{\sqrt{\sin^3 x \cos^5 x}} dx$$

$$(iv) \quad \int \frac{x^2 + 1}{x^4 + 1} dx$$

$$(v) \quad \int \frac{1}{\sqrt{(x-a)(x-b)}} dx$$

$$(vi) \quad \int \frac{5x-2}{3x^2+2x+1} dx$$

$$(vii) \quad \int \frac{x^2}{x^2+6x+1} dx$$

$$(viii) \quad \int \frac{x+2}{\sqrt{4x-x^2}} dx$$

$$(ix) \quad \int x \sqrt{1+x-x^2} dx$$

$$(x) \quad \int \frac{\sin^4 x}{\cos^8 x} dx$$

$$(xi) \quad \int \sqrt{\sec x - 1} dx \text{ [Hint: Multiply and divided by } \sqrt{\sec x + 1}]$$

Evaluate :

$$3. \quad (I) \quad \int \frac{dx}{x(x^7+1)}$$

$$(II) \quad \int \frac{3x+5}{x^3-x^2-x+1} dx$$

$$(III) \quad \int \frac{\sin \theta \cos \theta}{\cos^2 \theta - \cos \theta - 2} d\theta$$

$$(iv) \quad \int \frac{dx}{(2-x)(x^2+3)}$$

$$(v) \quad \int \frac{x^2+x+2}{(x-2)(x-1)} dx$$

$$(vi) \quad \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$$

$$(vii) \quad \int \frac{dx}{(2x+1)(x^2+4)}$$

$$(viii) \quad \int \frac{x^2-1}{x^4+x^2+1} dx$$

$$(ix) \quad \int \sqrt{\tan x} dx$$

$$(x) \quad \int \frac{dx}{\sin x - \sin 2x}$$

4. Evaluate:

$$(I) \quad \int x^5 \sin x^3 dx$$

$$(II) \quad \int \sec^3 x dx$$

$$(III) \quad \int e^{ax} \cos(bx + c) dx$$

$$(IV) \quad \int \sin^{-1} \left( \frac{6x}{1+9x^2} \right) dx \quad \text{[Hint: Put } 3x = \tan \theta \text{]}$$

$$(V) \quad \int \cos \sqrt{x} dx$$

$$(VI) \quad \int x^3 \tan^{-1} x dx$$

$$(VII) \quad \int e^{2x} \left( \frac{1+\sin 2x}{1+\cos 2x} \right) dx$$

$$(VIII) \quad \int \left[ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx$$

$$(IX) \quad \int \sqrt{2ax - x^2} dx$$

$$(X) \quad \int e^x \frac{(x^2+1)}{(x+1)^2} dx$$

$$(XI) \quad \int x^3 \sin^{-1} \left( \frac{1}{x} \right) dx$$

$$(XII) \quad \int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$$

[Hint: Put  $\log x = t$   
 $x = e^t$ ]

$$(XIII) \quad \int (6x + 5) \sqrt{6 + x - x^2} dx$$

$$(XIV) \quad \int \frac{1}{x^3 + 1} dx$$

$$(XV) \quad \int \tan^{-1} \left( \frac{x-5}{1+5x} \right) dx$$

$$(XVI) \quad \int \frac{dx}{5+4 \cos x}$$

5. Evaluate the following definite integrals:

$$(i) \quad \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

$$(ii) \quad \int_0^{\pi/2} \cos 2x \log \sin x dx$$

$$(iii) \quad \int_0^1 x \sqrt{\frac{1-x^2}{1+x^2}} dx$$

$$(iv) \quad \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$(v) \quad \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$$

$$(vi) \quad \int_0^1 \sin \left( 2 \tan^{-1} \sqrt{\frac{1+x}{1-x}} \right) dx$$

$$(vii) \quad \int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x} dx$$

$$(viii) \int_0^1 x \log \left( 1 + \frac{x}{2} \right) dx$$

$$(ix) \int_{-1}^{1/2} |x \cos \pi x| dx$$

$$(x) \int_{-\pi}^{\pi} (\cos a x - \sin b x)^2 dx$$

6. Evaluate:

$$(i) \int_2^5 [|x - 2| + |x - 3| + |x - 4|] dx$$

$$(ii) \int_0^{\pi} \frac{x}{1 + \sin x} dx$$

$$(iii) \int_{-1}^1 e^{\tan^{-1} x} \left[ \frac{1+x+x^2}{1+x^2} \right] dx$$

$$(iv) \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$(v) \int_0^2 [x^2] dx$$

$$(vi) \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$(vii) \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \text{ [Hint: use } \int_0^a f(x) dx = \int_0^a f(a-x) dx]$$

7. Evaluate the following integrals:

$$(i) \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$$

$$(ii) \int_{-\pi/2}^{\pi/2} (\sin |x| + \cos |x|) dx$$

$$(iii) \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$

$$(iv) \int_0^{\pi} \frac{x \tan x}{\sec x + \operatorname{cosec} x} dx$$

$$(v) \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$$

8. Evaluate

$$(i) \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx \quad x \in [0, 1]$$

$$(ii) \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$$

$$(iii) \int \frac{x^2 e^x}{(x+z)^2} dx$$

$$(iv) \int \frac{x^2}{(x \sin x + \cos x)^2} dx$$

$$(v) \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$$

$$(vi) \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$(vii) \int \frac{\sin x}{\sin 4x} dx$$



$$(viii) \int_{-1}^{3/2} |x \sin \pi x| \, dx$$

$$(ix) \int \frac{\sin(x-a)}{\sin(x+a)} \, dx$$

$$(x) \int \frac{x^2}{(x^2+4)(x^2+9)} \, dx$$

$$(xi) \int \frac{\cos 5x + \cos 4x}{1 - 2 \cos 3x} \, dx$$

### SIX MARK QUESTIONS

9. Evaluate the following integrals:

$$(i) \int \frac{x^5 + 4}{x^5 - x} \, dx$$

$$(ii) \int \frac{2e^t}{e^{3t} - 6e^{2t} + 11e^t - 6} \, dt$$

$$(iii) \int \frac{2x^3}{(x+1)(x-3)^2} \, dx$$

$$(iv) \int \frac{1 + \sin x}{\sin x (1 + \cos x)} \, dx$$

$$(v) \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) \, dx$$

$$(vi) \int_0^1 x \sqrt{\frac{1-x^2}{1+x^2}} \, dx$$

$$(vii) \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx$$

10. Evaluate the following integrals as limit of sums:

$$(i) \int_2^4 (2x + 1) dx$$

$$(ii) \int_0^2 (x^2 + 3) dx$$

$$(iii) \int_1^3 (3x^2 - 2x + 4) dx$$

$$(iv) \int_0^4 (3x^2 + e^{2x}) dx$$

$$(v) \int_0^1 e^{2-3x} dx$$

$$(vi) \int_0^1 (3x^2 + 2x + 1) dx$$

11. Evaluate:

$$(i) \int \frac{dx}{(\sin x - 2 \cos x)(2 \sin x + \cos x)}$$

$$(ii) \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

$$(iii) \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x) dx$$

$$12. \int_0^1 x(\tan^{-1} x)^2 dx$$

$$13. \int_0^{\pi/2} \log \sin x dx$$

$$14. \text{ Prove that } \int_0^1 \tan^{-1} \left( \frac{1}{1-x+x^2} \right) dx = 2 \int_0^1 \tan^{-1} x dx$$

Hence or otherwise evaluate the integral  $\int \tan^{-1}(1-x+x^2) dx$ .

$$15. \text{ Evaluate } \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx.$$

### Answers

#### ONE MARK QUESTIONS

$$1. \frac{\pi}{2} x + c$$

$$9. 0$$

$$2. 2e - 2$$

$$10. \frac{(x-2)\sqrt{x^2-4x+10}}{2} +$$

$$3. \tan x + c$$

$$3 \log|(x-2) +$$

$$4. 0$$

$$\sqrt{x^2-4x+10}| + c$$

$$5. \log|\log|\log x|| + c$$

$$11. 0$$

$$6. 0$$

$$12. \tan x - \cot x + c$$

$$7. \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$$

$$13. 3 \log_e 2$$

$$8. \tan x + c$$

$$14. \log|x| + c$$

$$15. \frac{\left(\frac{e}{a}\right)^x}{\log\left(\frac{e}{a}\right)} + c$$

$$16. \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + c$$

$$17. \log|x+1| + \frac{1}{x+1} + c$$

$$18. 2e^{\sqrt{x}} + c$$

$$19. x\cos^2\alpha + c$$

$$20. \frac{\log|x\cos\alpha+1|}{\cos\alpha} + c$$

$$21. \frac{(\log|\sec x + \tan x|)^2}{2} + c$$

$$22. \frac{\log|\cos\alpha + x\sin\alpha|}{\sin\alpha} + c$$

$$23. \tan|\log x| + c$$

$$24. \log|e^x + \sqrt{4 + e^{2x}}| + c$$

$$25. \frac{1}{3}\log|2 + 3\log x| + c$$

$$26. \log|x + \cos x| + c$$

$$27. 2\log\left|\sec\frac{x}{2}\right| + c$$

$$28. \frac{1}{e}\log|x^e + e^x| + c$$

$$29. \frac{(x+\log x)^2}{2} + c$$

$$30. 2$$

$$31. 1$$

$$32. \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$$

$$33. \frac{b-a}{2}$$

$$34. -1$$

$$35. 0$$

$$36. \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$$

$$37. 0$$

$$38. \frac{1}{2}\log|\sec x + \tan x| + c$$

$$39. 2-\sqrt{2}$$

$$40. \log|1 + \sin x| + c$$

$$41. x - \sin x + c$$

$$42. \log|\cos x + \sin x| + c$$

## TWO MARK QUESTIONS

$$1. x + \log x + c$$

2.  $\frac{2}{3} \left[ (x+2)^{3/2} - (x+1)^{3/2} \right] + c$
3.  $\frac{-1}{2} \left[ \frac{\sin 3x}{3} - \sin x \right] + c$
4.  $\frac{1}{a} \frac{x^2}{2} + a \log |x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$
5. 0
6.  $\frac{\left(\frac{a}{c}\right)^x + \left(\frac{b}{c}\right)^x}{\log \left| \frac{a}{c} \right| + \log \left| \frac{b}{c} \right|} + c$
7.  $\frac{ax^2}{2} + \frac{\log |x|}{a} - 2x + c$
8.  $\frac{2^x e^x}{\log(2e)} + c$
9.  $\frac{2^{2^{2^x}}}{(\log^2)^3} + C$
10.  $\frac{-\left[\cos 2(\tan^{-1} x)\right]}{2} + C$
11.  $\frac{x^2}{2} \log 2x - \frac{x^2}{4} + C$
12. 1
13. 1
14.  $\frac{19}{99}$
15.  $\tan^{-1} e - \frac{\pi}{4}$
16.  $\frac{\log |\sin x|^2}{2} + C$
17.  $\log |\sec x + \tan x| + \log |\operatorname{cosec} x - \cot x| + C$
18.  $\frac{2}{3} (\tan x)^{3/2} + C$
19.  $-\frac{2}{b^2} \left[ \log |a + b \cos x| + \frac{a}{a + b \cos x} \right] + C$
20.  $x - \frac{1}{2} \log |x^2 + 1| + \tan^{-1} x + C$

### FOUR MARK QUESTIONS

1. (I)  $\frac{1}{2} \log \left[ \operatorname{cosec}(\tan^{-1} x^2) - \frac{1}{x^2} \right] + c$
- (II)  $\frac{1}{2} (x^2 - x\sqrt{x^2 - 1}) + \frac{1}{2} \log |x + \sqrt{x^2 - 1}| + c$
- (III)  $\frac{1}{\sin(a-b)} \log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| + c$
- (IV)  $x \cos 2a - \sin 2a \log |\sec(x-a)| + c$
- (V)  $\frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$

$$(VI) \quad \frac{1}{5} \log |\sec 5x| - \frac{1}{2} \log |\sec 2x| - \frac{1}{3} \log |\sec 3x| + c$$

$$(VII) \quad \frac{1}{32} \left[ 2x + \frac{1}{2} \sin 2x - \frac{1}{2} \sin 4x - \frac{1}{6} \sin 6x \right] + c$$

$$(VIII) \quad - \left( \frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} \right) + c$$

$$(IX) \quad \frac{1}{a^2 - b^2} \sqrt{a^2 \sin^2 x + b^2 \cos^2 x} + c$$

$$(X) \quad -2 \operatorname{cosec} a \sqrt{\cos a - \tan x \sin a} + c$$

$$(XI) \quad \tan x - \cot x - 3x + c$$

$$(XII) \quad \sin^{-1}[\sin x - \cos x] + c$$

$$2. \quad (I) \quad \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x^2 + 1}{\sqrt{3}} \right) + c$$

$$(II) \quad \log \left| \frac{2 \log x}{3 \log x} \right| + c$$

$$(III) \quad \frac{-2}{\sqrt{\tan x}} + \frac{2}{3} \tan^{3/2} x + c$$

$$(IV) \quad \frac{1}{\sqrt{2}} \tan^{-1} \left\{ \frac{1}{\sqrt{2}} \left( x - \frac{1}{x} \right) \right\} + c$$

$$(V) \quad 2 \log |\sqrt{x-a} + \sqrt{x-b}| + c$$

$$(VI) \quad \frac{5}{6} \log |3x^2 + 2x + 1| + \frac{-11}{3\sqrt{2}} \tan^{-1} \left( \frac{3x+1}{\sqrt{2}} \right) + c$$

$$(VII) \quad x - 3 \log |x^2 + 6x + 12| + 2\sqrt{3} \tan^{-1} \left( \frac{x+3}{\sqrt{3}} \right) + c$$

$$(VIII) \quad -\sqrt{4x - x^2} + 4 \sin^{-1} \left( \frac{x-2}{2} \right) + c$$

$$(IX) \quad -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2} + \frac{5}{16} \sin^{-1} \left( \frac{2x-1}{\sqrt{5}} \right) + c$$

$$(X) \quad \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + c$$

$$(XI) \quad -\log \left| \cos x + \frac{1}{2} + \sqrt{\cos^2 x + \cos x} \right| + c$$

$$3. \quad (I) \quad \frac{1}{7} \log \left| \frac{x^7}{x^7+1} \right| + c$$

$$(II) \quad \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{x-1} + c$$

$$(III) \quad \frac{-2}{3} \log |\cos \theta - 2| - \frac{1}{3} \log |1 + \cos \theta| + c$$

$$(IV) \quad \frac{1}{14} \log \left| \frac{x^2+3}{(2-x)^2} \right| + \frac{2}{7\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) + c$$

$$(V) \quad x + 4 \log \left| \frac{(x-2)^2}{x-1} \right| + c$$

$$(VI) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) - 3 \tan^{-1} \left( \frac{x}{2} \right) + c$$

$$(VII) \quad \frac{2}{17} \log |2x+1| - \frac{1}{17} \log |x^2+4| + \frac{1}{34} \tan^{-1} \frac{x}{2} + c$$

$$(VIII) \quad \frac{1}{2} \log \left| \frac{x^2-x+1}{x^2+x+1} \right| + c$$

$$(IX) \quad \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right| + c$$

$$(X) -\frac{1}{2}\log|\cos x - 1| - \frac{1}{6}\log|\cos x + 1| + \frac{2}{3}\log|1 - 2\cos x| + c$$

$$4. (i) \quad \frac{1}{3}[-x^3 \cos x^3 + \sin x^3] + c$$

$$(ii) \quad \frac{1}{2}[\sec x \tan x + \log|\sec x + \tan x|] + c$$

$$(iii) \quad \frac{e^{ax}}{a^2+b^2} [a \cos(bx + c) + b \sin(bx + c)] + c$$

$$(iv) \quad 2x \tan^{-1} 3x - \frac{1}{3} \log |1 + 9x^2| + c$$

$$(v) \quad 2[\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}] + c$$

$$(vi) \quad \left(\frac{x^4-1}{4}\right) \tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} + c$$

$$(vii) \quad \frac{1}{2} e^{2x} \tan x + c$$

$$(viii) \quad \frac{x}{\log x} + c$$

$$(ix) \quad \left(\frac{x-a}{2}\right) \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a}\right) + c$$

$$(x) \quad e^x \left(\frac{x-1}{x+1}\right) + c$$

$$(xi) \quad \frac{x^4}{4} \sin^{-1} \left(\frac{1}{x}\right) + \frac{x^2+2}{12} \sqrt{x^2 - 1} + c$$

$$(xii) \quad x \log|\log x| - \frac{x}{\log x} + c$$

$$(xiii) \quad -2(6 + x - x^2)^{\frac{3}{2}} + 8 \left[ \frac{2x-1}{4} \sqrt{6 + x - x^2} + \frac{25}{8} \sin^{-1} \left(\frac{2x-1}{5}\right) \right] + c$$

$$(xiv) \quad \frac{1}{3} \log|x + 1| - \frac{1}{6} \log|x^2 - x + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}}\right) + c$$

$$(xv) \quad x \tan^{-1} x - \frac{1}{2} \log|1 + x^2| - x \tan^{-1} 5 + c$$

$$(xvi) \quad \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2}\right) + c$$



5. (I)  $\frac{1}{20} \log 3$   
 (II)  $-\pi/4$   
 (III)  $\frac{\pi}{4} - \frac{1}{2}$   
 (IV)  $\frac{\pi}{4} - \frac{1}{2} \log 2$   
 (V)  $\frac{\pi}{2}$   
 (VI)  $\pi/4$   
 (VII)  $\pi/2$   
 (VIII)  $\frac{3}{4} + \frac{3}{2} \log \frac{2}{3}$   
 (IX)  $\frac{3}{2\pi} - \frac{1}{\pi^2}$   
 (X)  $2\pi + \frac{1}{2a} \sin 2a\pi - \frac{1}{2b} \sin 2b\pi$
6. (I)  $\frac{1}{2}$   
 (II)  $\pi$   
 (III)  $e^{\pi/4} + e^{-\pi/4}$   
 (IV)  $\frac{1}{4} \pi^2$   
 (V)  $5 - \sqrt{3} - \sqrt{2}$   
 (VI)  $\frac{\pi^2}{16}$   
 (VII)  $\frac{\pi^2}{2a}$

7. (I)  $\frac{\pi}{12}$

(II) 2

(III)  $\frac{\pi}{2}$

(IV)  $\frac{\pi^2}{4}$

(V)  $a\pi$

8. (I)  $\frac{2(2x-1)}{\pi} \sin^{-1} \sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + c$

(II)  $-2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x-x^2} + c$

(III)  $\frac{x-2}{x+2} e^x + c$

(IV)  $\frac{\sin x - x \cos x}{x \sin x + \cos x} + c$

(V)  $(x+a) \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{ax} + c$

(VI)  $2 \sin^{-1} \frac{\sqrt{3}-1}{2}$

(VII)  $\frac{1}{8} \log \left| \frac{1-\sin x}{1+\sin x} \right| - \frac{1}{4\sqrt{2}} \log \left| \frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x} \right| + c$

(VIII)  $\frac{3}{\pi} + \frac{1}{\pi^2}$

(IX)  $(\cos 2a)(x+a) - (\sin 2a) \log |\sin(x+a)| + c$

(X)  $-\frac{4}{5} \log |x^2+4| + \frac{9}{5} \log |x^2+9| + c$

(XI)  $-\left(\frac{1}{2} \sin 2x + \sin x\right) + c$

$$9. \quad (I) \quad x - 4 \log|x| + \frac{5}{4} \log|x-1| + \frac{3}{4} \log|x+1| + \log|x^2 + 1| - \frac{1}{2} \tan^{-1} x + c$$

$$(II) \quad \log \left| \frac{(e^t-1)(e^t-3)}{(e^t-2)^2} \right| + c$$

$$(III) \quad 2x - \frac{1}{8} \log|x+1| + \frac{81}{8} \log|x-3| - \frac{27}{2(x-3)} + c$$

$$(IV) \quad \frac{1}{4} \log \left| \frac{1-\cos x}{1+\cos x} \right| + \frac{1}{2(1+\cos x)} + \tan \frac{x}{2} + c$$

$$(V) \quad \frac{\pi}{\sqrt{2}}$$

$$(VI) \quad \frac{\pi-2}{4}$$

$$(I) \quad \frac{\pi}{4} - \frac{1}{2} \log 2$$

$$10. \quad (I) \quad 14$$

$$(II) \quad \frac{26}{3}$$

$$(III) \quad 26$$

$$(IV) \quad \frac{1}{2}(127 + e^8)$$

$$(V) \quad \frac{1}{3} \left( e^2 - \frac{1}{e} \right)$$

$$(VI) \quad 3$$

$$11. \quad (I) \quad \frac{1}{5} \log \left| \frac{\tan x - 2}{2 \tan x + 1} \right| + c$$

$$(II) \quad \frac{\pi}{8} \log 2$$

$$(III) \quad \frac{\pi}{2} \log \frac{1}{2}$$

$$12. \quad \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2$$

$$13. \quad \frac{-\pi}{2} \log 2$$

$$14. \quad \log 2$$

$$15. \quad \frac{1}{\sqrt{2}} \log |\sqrt{2} + 1|$$

## CHAPTER 8

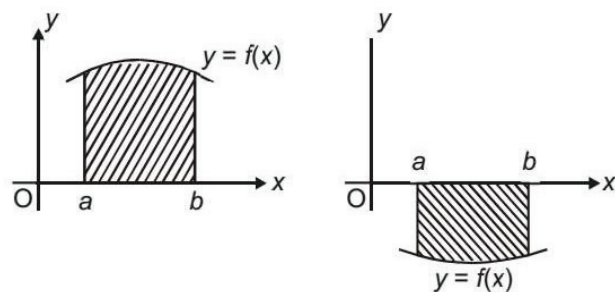
### APPLICATIONS OF INTEGRALS

#### POINT TO REMEMBER

##### AREA OF BOUNDED REGION

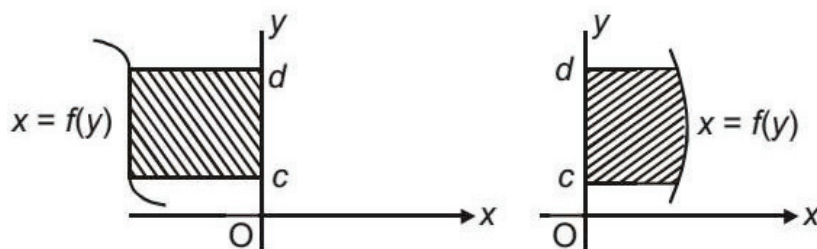
- Area bounded by the curve  $y = f(x)$ , the  $x$  axis and between the ordinates,  $x = a$  and  $x = b$  is given by

$$Area = \left| \int_a^b f(x) dx \right|$$

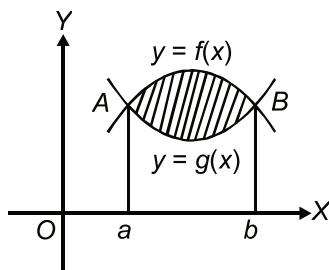


- Area bounded by the curve  $x = f(y)$ , the  $y$ -axis and between the abscissas,  $y = c$  and  $y = d$  is given by

$$Area = \left| \int_c^d f(y) dy \right|$$

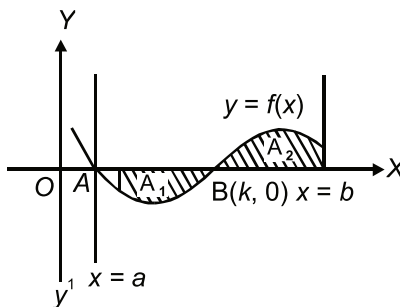


- Area bounded by two curves  $y = f(x)$  and  $y = g(x)$  such that  $0 \leq g(x) \leq f(x)$  for all  $x \in [a, b]$  and between the ordinates  $x = a$  and  $x = b$  is given by



$$\text{Area} = \int_a^b [f(x) - g(x)] dx$$

- Area of the following shaded region =  $\left| \int_a^k f(x) dx \right| + \int_k^b f(x) dx$



### FOUR/SIX MARK QUESTIONS

- Find the area of the parabola  $y^2 = 4ax$  bounded by its Latus rectum.
- Find the area of the region  $\{(x, y): x^2 \leq y \leq |x|\}$ .
- Find the area of region in the first quadrant enclosed by  $x$ -axis, the line  $y = x$  and the circle  $x^2 + y^2 = 32$ .

4. Find the area of region  $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$
5. Prove that the curve  $y = x^2$  and  $x = y^2$  divide the square bounded by  $x = 0, y = 0, x = 1, y = 1$  into three equal parts.
6. Find the area of the smaller region enclosed between ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  and the line  $bx + ay = ab$ .
7. Find the common area bounded by the circles  $x^2 + y^2 = 4$  and  $(x - 2)^2 + y^2 = 4$ .
8. Using integration, find the area of the triangle whose sides are given by  $2x + y = 4, 3x - 2y = 6$  and  $x - 3y + 5 = 0$ .
9. Using integration, find the area of the triangle whose vertices are  $(-1, 0), (1, 3)$  and  $(3, 2)$ .
10. Find the area of the region  $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$ .
11. Find the area of the region bounded by the curve  $x^2 = 4y$  and the line  $x = 4y - 2$ .
12. Find the area lying above x-axis and included between the circle  $x^2 + y^2 = 8x$  and inside the parabola  $y^2 = 4x$ .
13. Using integration, find the area enclosed by the curve  $y = \cos x, y = \sin x$  and x-axis in the interval  $[0, \pi/2]$ .
14. Using integration, find the area of the following region:  
 $\{(x, y) : |x - 1| \leq y \leq \sqrt{5 - x^2}\}$
15. Using integration, find the area of the triangle formed by positive x-axis and tangent and normal to the circle  $x^2 + y^2 = 4$  at  $(1, \sqrt{3})$ .
16. Using integration, find the area of the region bounded by the line  $x - y + 2 = 0$ , the curve  $x = \sqrt{y}$  and y-axis.
17. Find the area of the region bounded by the curves  $ay^2 = x^3$ , the y-axis and the lines  $y = a$  and  $y = 2a$ .

18. Find the area bounded by x-axis, the curve  $y = 2x^2$  and tangent to the curve at the point whose abscissa is 2.
19. Using integration, find the area of the region bounded by the curve  $y = 1 + |x + 1|$  and lines  $x = -3, x = 3, y = 0$ .
20. Find the area of the region  $\{(x, y): y^2 \geq 6x, x^2 + y^2 \leq 16\}$
21. Find the area of the region enclosed between curves  $y = |x - 1|$  and  $y = 3 - |x|$ .
22. If the area bounded by the parabola  $y^2 = 16ax$  and the line  $y = 4mx$  is  $\frac{a^2}{12}$  sq unit then using integration find the value of m.
23. Given  $\frac{dy}{dx}$  is directly proportional to the square of x and  $\frac{dy}{dx} = 6$  at  $x = 2$ . Then find the equation of the curve, when  $x = 2$  and  $y = 4$ . Also find the area of the region bounded by curve between lines  $y = 1$  and  $y = 3$ .
24. Find the area between x-axis, curve  $x = y^2$  and its normal at the point (1,1).
25. Find the area of the region bounded by the curves  $x = at^2$  and  $y = 2at$  between the ordinates corresponding to  $t = 1$  and  $t = 2$
26. Using integration find the area bounded by the tangent to the curve  $y = 3x^2$  at the point (1, 3), and the Lines whose equations are  $y = \frac{x}{3}$  and  $x + y = 4$

## ANSWERS

- |  |  |
|--|--|
| 1. $\frac{8}{3}a^2$ sq. units  | 9. 4 sq. units   |
| 2. $\frac{1}{3}$ sq. units   | 10. $\left(\pi - \frac{1}{2}\right)$ sq. units                     |
| 3. $4\pi$ sq. units  | 11. $\frac{9}{8}$ sq. units  |
| 4. $\left[\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$ sq. units | 12. $\frac{4}{3}(8 + 3\pi)$ sq. units                              |
| 5. ....  | 13. $(2 - \sqrt{2})$ sq. units                                     |
| 6. $\left(\frac{\pi-2}{4}\right)ab$ sq. units  | 14. $\left(\frac{5\pi}{4} - \frac{1}{2}\right)$ sq. units          |
| 7. $\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$ sq. units   | 15. $2\sqrt{3}$ sq. units  |
| 8. 3.5 sq. units   | 16. $\frac{10}{3}$ sq. units                                       |
| 18. $\frac{4}{3}$ sq. units  | 17. $\frac{3}{5}a^2 \left[(32)^{\frac{1}{3}} - 1\right]$ sq. units |
| 19. 16 sq. units   | 20. $\frac{32\pi - 4\sqrt{3}}{3}$ sq. units                        |
| 22. $m = 2$  | 21. 4 sq. units  |
| 24. $\frac{11}{2}$ Square units  | 23. $\frac{3}{4}(2)^{1/3} \left[(3)^{4/3} - 1\right]$ sq units     |
| 26. $\frac{56}{17}$ Square units   | 25. $\frac{56a^2}{3}$ Square units                                 |



## CHAPTER–9

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### DIFFERENTIAL EQUATIONS

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#### POINTS TO REMEMBER

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- **Differential Equation:** Equation containing derivatives of a dependent variable with respect to an independent variable is called differential equation.
- **Order of a Differential Equation:** The order of a differential equation is defined to be the order of the highest order derivative occurring in the differential equation.
- **Degree of a Differential Equation:** Highest power of highest order derivative involved in the equation is called degree of differential equation where equation is a polynomial equation in differential coefficients.
- **Formation of a Differential Equation:** We differentiate the family of curves as many times as the number of arbitrary constant in the given family of curves. Now eliminate the arbitrary constants from these equations.

After elimination, the equation obtained is differential equation.

- **Solution of Differential Equation**

- (i) **Variable Separable Method**

$$\frac{dy}{dx} = f(x, y).$$

We Separate the variables and get

$$f(x)dx = g(y)dy$$

Then  $\int f(x)dx = \int g(y)dy + c$  is the required solutions.

- (ii) **Homogeneous Differential Equation:** A differential equation of the form  $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$  where  $f(x, y)$  and  $g(x, y)$  are both homogeneous functions of the same degree in  $x$  and  $y$  i.e., of the form  $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$  is called a homogeneous differential equation.

For solving this type of equations we substitute  $y = vx$  and then  $\frac{dy}{dx} = v + x \frac{dv}{dx}$ . The equation can be solved by variables separable method.

A homogeneous differential equation can be of the form  $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$

To solve this equation, we substitute  $x = vy$  and then  $\frac{dx}{dy} = v + y \frac{dv}{dy}$  then the equation can be solved by variable separate method.

- (iii) **Linear Differential Equation:** An equation of the form  $\frac{dy}{dx} + Py = Q$

where  $P$  and  $Q$  are constant or functions of  $x$  only is called a linear differential equation. For finding solution of this type of equations, we find integrating factor (I.F.) =  $e^{\int P dx}$

Solution is  $y (I.F.) = \int Q \cdot (I.F.) dx + c$

Similarly, differential equations of the type  $\frac{dx}{dy} + Px = Q$  where  $P$  and  $Q$  are constants or functions of  $y$  only can be solved.

Here, I.F. =  $e^{\int P dy}$  and the solution is  $x (I.F.) = \int Q \times (I.F.) dy + C$

## ONE MARK QUESTIONS

- Write the order and degree of the following differential equations.

$$(i) \quad \frac{dy}{dx} + \cos y = 0$$

$$(ii) \quad \left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} = 4$$

$$(iii) \quad \frac{d^4y}{dx^4} + \sin x = \left(\frac{d^2y}{dx^2}\right)^5$$

$$(iv) \quad \frac{d^5y}{dx^5} + \log\left(\frac{dy}{dx}\right) = 0$$

$$(v) \quad \sqrt{1 + \frac{dy}{dx}} = \left(\frac{d^2y}{dx^2}\right)^{1/3}$$

$$(vi) \quad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k \frac{d^2y}{dx^2}$$

$$(vii) \quad \left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = \sin x$$

$$(viii) \quad \frac{dy}{dx} + \tan\left(\frac{dy}{dx}\right) = 0$$

2. Write integrating factor differential equations :

$$(I) \quad \frac{dy}{dx} + y \cos x = \sin x$$

$$(II) \quad \frac{dy}{dx} + y \sec^2 x = \sec x + \tan x$$

$$(III) \quad x^2 \frac{dy}{dx} + y = x^4$$

$$(IV) \quad x \frac{dy}{dx} + y \log x = x + y$$

$$(V) \quad x \frac{dy}{dx} - 3y = x^3$$

$$(VI) \quad \frac{dy}{dx} + y \tan x = \sec x$$

$$(VII) \quad \frac{dy}{dx} + \frac{1}{1+x^2} y = \sin x$$

3. Write order of the differential equation of the family of following curves.

$$(I) \quad y = Ae^x + Be^{x+c}$$

$$(II) \quad Ay = Bx^2$$

$$(III) \quad (x - a)^2 + (y - b)^2 = 9$$

$$(IV) \quad Ax + By^2 = Bx^2 - Ay$$

$$(V) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$(VI) \quad y = a \cos (ax + b)$$

$$(VII) \quad y = a + be^{x+c}$$

### TWO MARK QUESTIONS

1. Write the general solution of the following differential equations.

$$(i) \quad \frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$$

$$(ii) \quad (e^x + e^{-x}) dy = (e^x - e^{-x}) dx$$

$$(iii) \quad \frac{dy}{dx} = x^3 + e^x + x^e$$

$$(iv) \quad \frac{dy}{dx} = 5^{x+y}$$

$$(v) \quad \frac{dy}{dx} = \frac{1 - \cos 2x}{1 + \cos 2y}$$

$$(vi) \quad \frac{dy}{dx} = \frac{1 - 2y}{3x + 1}$$

### FOUR MARK QUESTIONS

1. (I) Show that  $y = e^{m \sin^{-1} x}$  is a solution of

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0$$

- (II) Show that  $y = \sin(\sin x)$  is a solution of differential equation

$$\frac{d^2 y}{dx^2} + (\tan x) \frac{dy}{dx} + y \cos^2 x = 0$$

- (III) Show that  $y = Ax + \frac{B}{x}$  is a solution of  $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0$

- (IV) Show that  $y = a \cos(\log x) + b \sin(\log x)$  is a solution of

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

- (V) Verify that  $y = \log(x + \sqrt{x^2 + a^2})$  satisfies the differential

$$\text{equation: } (a^2 + x^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0$$

(VI) Find the differential equation of the family of curves

$$y = e^x(A \cos x + B \sin x), \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

(VII) Find the differential equation of an ellipse with major and minor axes  $2a$  and  $2b$  respectively.

(VIII) Form the differential equation representing the family of curves  $(y - b)^2 = 4(x - a)$ .

2. Solve the following differential equations.

(I)  $(1 - x^2) \frac{dy}{dx} - xy = x^2$ , given that  $x = 0, y = 2$

(II)  $x \frac{dy}{dx} + 2y = x^2 \log x$

(III)  $\frac{dy}{dx} + \frac{1}{x} y = \cos x + \frac{\sin x}{x}$ ,  $x > 0$

(IV)  $dy = \cos x (2 - y \operatorname{cosec} x) dx$ ; given that  $x = \frac{\pi}{2}, y = 2$

(V)  $y dx + (x - y^3) dy = 0$

(VI)  $ye^y dx = (y^3 + 2xe^y) dy$

3. Solve each of the following differential equations:

(I)  $y - x \frac{dy}{dx} = 2 \left( y^2 + \frac{dy}{dx} \right)$

(II)  $\cos y \, dx + (1 + 2e^{-x}) \sin y \, dy = 0$

$$(III) \quad x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$$

$$(IV) \quad \sqrt{(1-x^2)(1-y^2)}dy + xy dx = 0$$

$$(V) \quad (xy^2 + x)dx + (yx^2 + y)dy = 0; y(0) = 1$$

$$(VI) \quad \frac{dy}{dx} - y \sin^3 x \cos^3 x + xy e^x = 0$$

$$(VII) \quad \tan x \tan y dx + \sec^2 x \sec^2 y dy = 0$$

$$(VIII) \quad \frac{dy}{dx} = x - 1 + xy - y$$

4. Solve the following differential equations:

$$(I) \quad x^2 y dx - (x^3 + y^3) dy = 0$$

$$(II) \quad x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

$$(III) \quad (x^2 - y^2)dx + 2xy dy = 0, \quad y(1) = 1$$

$$(IV) \quad \left(y \sin \frac{x}{y}\right) dx = \left(x \sin \frac{x}{y} - y\right) dy$$

$$(V) \quad \frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

$$(VI) \quad x \frac{dy}{dx} = y (\log y - \log x + 1)$$

$$(VII) \quad \frac{dy}{dx} = e^{x+y} + x^2 e^y$$

$$(VIII) \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

$$(IX) \quad (3xy + y^2)dx + (x^2 + xy)dy = 0$$

5. (I) Form the differential equation of the family of circles touching y-axis at (0, 0).  
 (II) Form the differential equation of family of parabolas having vertex at (0,0) and axis along the (i) positive y-axis (ii) positive x-axis.  
 (III) Form differential equation of family of circles passing through origin and whose centres lie on x-axis.  
 (IV) Form the differential equation of the family of circles in the first quadrant and touching the coordinate axes.
6. Show that the differential equation  $\frac{dy}{dx} = \frac{x+2y}{x-y}$  is homogeneous and solve it.
7. Show that the differential equation:  
 $(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$  is homogeneous and solve it.
8. Solve the following differential equations:  
 (I)  $\frac{dy}{dx} - 2y = \cos 3x$   
 (II)  $\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$  if  $y\left(\frac{\pi}{2}\right) = 1$   
 (III)  $\log\left(\frac{dy}{dx}\right) = px + qy$
9. Solve the following differential equations:  
 (I)  $(x^3 + y^3) dx = (x^2y + xy^2)dy$



$$(II) \quad x \, dy - y \, dx = \sqrt{x^2 + y^2} \, dx$$

$$(III) \quad y \left\{ x \cos \left( \frac{y}{x} \right) + y \sin \left( \frac{y}{x} \right) \right\} dx$$

$$-x \left\{ y \sin \left( \frac{y}{x} \right) - x \cos \left( \frac{y}{x} \right) \right\} dy = 0$$

$$(IV) \quad x^2 dy + y(x + y) \, dx = 0 \text{ given that } y=1 \text{ when } x=1.$$

$$(V) \quad x e^{\frac{y}{x}} - y + x \frac{dy}{dx} = 0 \text{ if } y(e) = 0$$

$$(VI) \quad (x^3 - 3xy^2) \, dx = (y^3 - 3x^2y) \, dy$$

$$(VII) \quad \frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec} \left( \frac{y}{x} \right) = 0 \text{ given that } y = 0 \text{ when } x = 1$$

10. Solve the following differential equations:

$$(I) \quad \cos^2 x \frac{dy}{dx} = \tan x - y$$

$$(II) \quad x \cos x \frac{dy}{dx} + y (x \sin x + \cos x) = 1$$

$$(III) \quad \left( 1 + e^{\frac{x}{y}} \right) dx + e^{\frac{x}{y}} \left( 1 - \frac{x}{y} \right) dy = 0$$

$$(IV) \quad (y - \sin x) \, dx + \tan x \, dy = 0, \, y(0) = 0$$

### SIX MARK QUESTIONS

1 Solve the following differential equations:

$$(I) \quad (x \, dy - y \, dx)y \sin\left(\frac{y}{x}\right) = (y \, dx + x \, dy)x \cos\left(\frac{y}{x}\right)$$

$$(II) \quad 3e^x \tan y \, dx + (1 - e^x)\sec^2 y \, dy = 0 \text{ given that } y = \frac{\pi}{4}, \text{ when } x = 1$$

$$(III) \quad \frac{dy}{dx} + y \cot x = 2x + x^2 \cot x \text{ given that } y(0) = 0.$$

2. Show that the differential equation

$2y e^{\frac{x}{y}} dx + \left(y - 2x e^{\frac{x}{y}}\right) dy = 0$  is homogenous. Find the particular solution of this differential equation given that  $x = 0$  when  $y = 1$ .

### ANSWERS

#### ONE MARK QUESTIONS

1. (i) order = 1, degree is not defined

(ii) order = 2, degree = 1

(iii) order = 4, degree = 1

(iv) order = 5, degree is not defined.

(v) order = 2, degree = 2

(vi) order = 2, degree = 2

(vii) order = 3, degree = 2

(viii) order = 1, degree is not defined

2. (I)  $e^{\sin x}$

(II)  $e^{\tan x}$

(III)  $e^{-1/x}$

(IV)  $e^{\frac{(\log x)^2}{2}}$

(V)  $\frac{1}{x^3}$

(VI)  $\sec x$

(VII)  $e^{\tan^{-1} x}$

3. (I) 1 (II) 1  
 (III) 2 (IV) 1  
 (V) 1 (VI) 2  
 (VII) 2

### TWO MARK QUESTIONS

1. (I)  $y = \frac{x^6}{6} + \frac{x^3}{3} - 2 \log |x| + c$  (II)  $y = \log_e |e^x + e^{-x}| + c$   
 (III)  $y = \frac{x^4}{4} + e^x + \frac{x^{e+1}}{e+1} + c$  (IV)  $5^x + 5^{-y} = c$   
 (V)  $2(y - x) + \sin 2y + \sin 2x = c$   
 (VI)  $2 \log |3x + 1| + 3 \log |1 - 2y| = c$

### FOUR MARK QUESTIONS

1. (VI)  $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$   
 (VII)  $x \left( \frac{dy}{dx} \right)^2 + xy \frac{d^2y}{dx^2} = y \frac{dy}{dx}$   
 (VIII)  $2 \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^3 = 0$
2. (I)  $y\sqrt{1-x^2} + \frac{x\sqrt{1-x^2}}{2} = \frac{\sin^{-1} x}{2} + 2$   
 (II)  $y = \frac{x^2(4 \log_e x - 1)}{16} + \frac{c}{x^2}$   
 (III)  $y = \sin x + \frac{c}{x}, x > 0$   
 (IV)  $2y \sin x = 3 - \cos 2x$   
 (V)  $xy = \frac{y^4}{4} + c$   
 (VI)  $x = -y^2 e^{-y} + cy^2$

3. (I)  $cy = (x + 2)(1 - 2y)$
- (II)  $(e^x + 2) \sec y = c$
- (III)  $\sqrt{1 - x^2} + \sqrt{1 - y^2} = c$
- (IV)  $\frac{1}{2} \log \left| \frac{\sqrt{1-y^2}-1}{\sqrt{1-y^2}+1} \right| = \sqrt{1-x^2} - \sqrt{1-y^2} + c$
- (V)  $(x^2 + 1)(y^2 + 1) = 2$
- (VI)  $\log y = -\frac{1}{4} \cos^4 x + \frac{1}{6} \cos^6 x - xe^x + e^x + c$
- $$= \frac{1}{16} \left[ \frac{\cos^3 2x}{3} - \cos 2x \right] - (x - 1)e^x + c$$
- (VII)  $\log |\tan y| - \frac{\cos 2x}{4} = c$
- (VIII)  $\log |y + 1| = \frac{x^2}{2} - x + c$
4. (I)  $\frac{-x^3}{y^3} + 3 \log |y| = c$
- (II)  $\tan^{-1} \left( \frac{y}{x} \right) = \log |x| + c$
- (III)  $x^2 + y^2 = 2x$
- (IV)  $y = ce^{\cos(x/y)}$
- (V)  $\sin \left( \frac{y}{x} \right) = cx$
- (VI)  $\log |y/x| = cx$
- (VII)  $-e^{-y} = e^x + \frac{x^3}{3} + c$

$$(VIII) \quad \sin^{-1} y = \sin^{-1} x + c$$

$$(IX) \quad (2xy + y^2) = \frac{c}{x^2}$$

$$5. \quad (I) \quad x^2 - y^2 + 2xy \frac{dy}{dx} = 0$$

$$(II) \quad 2y = x \frac{dy}{dx}, \quad y = 2x \frac{dy}{dx}$$

$$(III) \quad x^2 - y^2 + 2xy \frac{dy}{dx} = 0$$

$$(IV) \quad (x - y)^2 (1 + y'^2) = (x + yy')^2$$

$$6. \quad \log|x^2 + xy + y^2| = 2\sqrt{3} \tan^{-1} \left( \frac{x+2y}{\sqrt{3x}} \right) + c$$

$$7. \quad x^2 + y^2 = c(x + y)$$

$$8. \quad (I) \quad y = \frac{3 \sin 3x}{13} - \frac{2 \cos 3}{13} + Ce^{2x}$$

$$(II) \quad y = \frac{2}{3} \sin^2 x + \frac{1}{3} \operatorname{cosec} x$$

$$(III) \quad \frac{e^{px}}{p} + \frac{e^{-qy}}{q} = c$$

$$9. \quad (I) \quad -y = x \log\{c(x - y)\}$$

$$(II) \quad cx^2 = y + \sqrt{x^2 + y^2}$$

$$(III) \quad xy \cos\left(\frac{y}{x}\right) = c$$

$$(IV) \quad 3x^2y = y + 2x$$

$$(V) \quad y = -x \log(\log|x|), \quad x \neq 0$$

$$(VI) \quad c(x^2 + y^2) = \sqrt{x^2 - y^2}$$

$$(VII) \quad \cos \frac{y}{x} = \log|x| + 1$$

$$10. \quad (I) \quad y = \tan x - 1 + ce^{-\tan x}$$

$$(II) \quad y = \frac{\sin x}{x} + c \frac{\cos x}{x}$$

$$(III) \quad x + ye^{\frac{x}{y}} = c$$

$$(IV) \quad 2y = \sin x$$

### SIX MARK QUESTIONS

$$1. \quad (I) \quad cxy = \sec\left(\frac{y}{x}\right)$$

$$(II) \quad (1 - e)^3 \tan y = (1 - e^x)^3$$

$$(III) \quad y = x^2$$

$$2. \quad e^{x/y} = -\frac{1}{2} \log|y| + 1$$

## CHAPTER–10

### VECTORS

#### POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a vector. It is denoted by a directed line segment.
- Two or more vectors which are parallel to same line are called collinear vectors.
- Position vector of a point  $P(a, b, c)$  w.r.t. origin  $(0, 0, 0)$  is denoted by  $\overrightarrow{OP}$  where  $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$  and  $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$ .
- If  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$  be any two points in space, then

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Any vector  $\vec{a}$  is called unit vector if  $|\vec{a}| = 1$ . It is denoted by  $\hat{a}$ .
- If two vectors  $\vec{a}$  and  $\vec{b}$  are represented in magnitude and direction by the two sides of a triangle in order, then their sum  $\vec{a} + \vec{b}$  is represented in magnitude and direction by third side of a triangle taken in opposite order. This is called triangle law of addition of vectors.
- If  $\vec{a}$  is any vector and  $\lambda$  is a scalar, then  $\lambda \vec{a}$  is vector collinear with  $\vec{a}$  and  $|\lambda \vec{a}| = |\lambda| |\vec{a}|$ .
- If  $\vec{a}$  and  $\vec{b}$  are two collinear vectors, then  $\vec{a} = \lambda \vec{b}$  where  $\lambda$  is some scalar.

- Any vector  $\vec{a}$  can be written as  $\vec{a} = |\vec{a}|\hat{a}$  where  $\hat{a}$  is a unit vector in the direction of  $\vec{a}$ .
- If  $\vec{a}$  and  $\vec{b}$  be the position vectors of points A and B, and C is any point which divides  $\overline{AB}$  in ratio m:n internally then position vector  $\vec{c}$  of point C is given as  $\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$ . If C divides  $\overline{AB}$  in ratio m:n externally, then  $\vec{c} = \frac{m\vec{b} - n\vec{a}}{m-n}$ .
- The angles  $\alpha, \beta$  and  $\gamma$  made by  $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$  with positive direction of x, y and z-axis are called angles and cosines of these angles are called direction cosines of  $\vec{r}$  usually denoted as  $l = \cos \alpha$ ,  $m = \cos \beta$ ,  $n = \cos \gamma$

Also  $l = \frac{a}{|\vec{r}|}$ ,  $m = \frac{b}{|\vec{r}|}$ ,  $n = \frac{c}{|\vec{r}|}$  and  $l^2 + m^2 + n^2 = 1$

- The numbers a, b, c proportional to l, m, n are called direction ratios.
- Scalar product or dot product of two vectors  $\vec{a}$  and  $\vec{b}$  is denoted as  $\vec{a} \cdot \vec{b}$  and is defined as  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ ,  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ . ( $0 \leq \theta \leq \pi$ ).
- Dot product of two vectors is commutative i.e.  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}, \vec{b} = \vec{0}$  or  $\vec{a} \perp \vec{b}$ .
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ , so  $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$
- If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ , then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$



- Projection of  $\vec{a}$  on  $\vec{b} = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}$  and

Projection vector of  $\vec{a}$  along  $\vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}$ .

- Cross product or vector product of two vectors  $\vec{a}$  and  $\vec{b}$  is denoted as  $\vec{a} \times \vec{b}$  and is defined as  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$ , where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$  ( $0 \leq \theta \leq \pi$ ). And  $\hat{n}$  is a unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$  such that  $\vec{a}$ ,  $\vec{b}$  and  $\hat{n}$  form a right handed system.
- Cross product of two vectors is not commutative i.e.,  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ , but  $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ .
- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}, \vec{b} = \vec{0}$  or  $\vec{a} \parallel \vec{b}$ .
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$ .
- $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$  and  $\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$
- If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ , then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both  $\vec{a}$  and  $\vec{b} = \pm \left( \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \right)$ .
- $|\vec{a} \times \vec{b}|$  is the area of parallelogram whose adjacent sides are  $\vec{a}$  and  $\vec{b}$
- $\frac{1}{2} |\vec{a} \times \vec{b}|$  is the area of parallelogram where diagonals are  $\vec{a}$  and  $\vec{b}$ .
- If  $\vec{a}, \vec{b}$  and  $\vec{c}$  form a triangle, then area of the triangle

- $= \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |\vec{b} \times \vec{c}| = \frac{1}{2} |\vec{c} \times \vec{a}|.$
- Scalar triple product of three vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$  is defined as  $\vec{a} \cdot (\vec{b} \times \vec{c})$  and is denoted as  $[\vec{a} \vec{b} \vec{c}]$
- Geometrically, absolute value of scalar triple product  $[\vec{a} \vec{b} \vec{c}]$  represents volume of a parallelepiped whose coterminal edges are  $\vec{a}, \vec{b}$  and  $\vec{c}$ .
- $\vec{a}, \vec{b}, \vec{c}$  are coplanar  $\Leftrightarrow [\vec{a} \vec{b} \vec{c}] = 0$
- $[\vec{a} \vec{b} \vec{c}] = [\vec{b} \vec{c} \vec{a}] = [\vec{c} \vec{a} \vec{b}]$
- If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  and  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$  then
 
$$[\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
- Then scalar triple product of three vectors is zero if any two of them are same or collinear.

### ONE MARK QUESTIONS

1. If  $\vec{AB} = 3\hat{i} + 2\hat{j} - \hat{k}$  and the coordinate of A are (4,1,1), then find the coordinates of B.
2. Let  $\vec{a} = -2\hat{i} + \hat{j}, \vec{b} = \hat{i} + 2\hat{j}$  and  $\vec{c} = 4\hat{i} + 3\hat{j}$ . Find the values of x and y such that  $\vec{c} = x\vec{a} + y\vec{b}$ .
3. Find a unit vector in the direction of the resultant of the vectors  $\hat{i} - \hat{j} + 3\hat{k}, 2\hat{i} + \hat{j} - 2\hat{k}$  and  $\hat{i} + 2\hat{j} - 2\hat{k}$ .

4. Find a vector of magnitude of 5 units parallel to the resultant of vector  $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$  and  $\vec{b} = (\hat{i} - 2\hat{j} - \hat{k})$
5. For what value of  $\lambda$  are the vector  $\vec{a}$  and  $\vec{b}$  perpendicular to each other?  
Where  $\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$  and  $\vec{b} = 5\hat{i} - 9\hat{j} + 2\hat{k}$
6. Write the value of p for which  $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$  and  $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$  are parallel vectors.
7. For any two vectors  $\vec{a}$  and  $\vec{b}$  write when  $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$  holds.
8. Find the value of p if  $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + 3\hat{j} + p\hat{k}) = \vec{0}$
9. Evaluate:  $\hat{i} \cdot (\hat{j} \times \hat{k}) + (\hat{i} \times \hat{k}) \cdot \hat{j}$
10. If  $\vec{a} = 2\hat{i} - 3\hat{j}$ ,  $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ ,  $\vec{c} = 3\hat{i} - \hat{k}$ , find  $[\vec{a}\vec{b}\vec{c}]$
11. If  $\vec{a} = 5\hat{i} - 4\hat{j} + \hat{k}$ ,  $\vec{b} = -4\hat{i} + 3\hat{j} - 2\hat{k}$  and  $\vec{c} = \hat{i} - 2\hat{j} - 2\hat{k}$ , then evaluate  $\vec{c} \cdot (\vec{a} \times \vec{b})$
12. Show that vector  $\hat{i} + 3\hat{j} + \hat{k}$ ,  $2\hat{i} - \hat{j} - \hat{k}$ ,  $7\hat{j} + 3\hat{k}$  are parallel to same plane.
13. Find a vector of magnitude 6 which is perpendicular to both the vectors  $2\hat{i} - \hat{j} + 2\hat{k}$  and  $4\hat{i} - \hat{j} + 3\hat{k}$ .
14. If  $\vec{a} \cdot \vec{b} = 0$ , then what can you say about  $\vec{a}$  and  $\vec{b}$ ?
15. If  $\vec{a}$  and  $\vec{b}$  are two vectors such that  $|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$ , then what is the angle between  $\vec{a}$  and  $\vec{b}$ ?

16. Find the area of a parallelogram having diagonals  $3\hat{i} + \hat{j} - 2\hat{k}$  and  $\hat{i} - 3\hat{j} + 4\hat{k}$ .
17. If  $\hat{i}, \hat{j}$  and  $\hat{k}$  are three mutually perpendicular vectors, then find the value of  $\hat{j} \cdot (\hat{k} \times \hat{i})$ .
18. P and Q are two points with position vectors  $3\vec{a} - 2\vec{b}$  and  $\vec{a} + \vec{b}$  respectively. Write the position vector of a point R which divides the segment PQ in the ratio 2:1 externally.
19. Find  $\lambda$  when scalar projection of  $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$  on  $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$  is 4 units.
20. Find "a" so that the vectors  $\vec{p} = 3\hat{i} - 2\hat{j}$  and  $\vec{q} = 2\hat{i} + a\hat{j}$  be orthogonal.
21. If  $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ ,  $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$  and  $\vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$  are coplanar, find the value of  $\lambda$ .
22. What is the point of trisection of PQ nearer to P if positions of P and Q are  $3\hat{i} + 3\hat{j} - 4\hat{k}$  and  $9\hat{i} + 8\hat{j} - 10\hat{k}$  respectively?
23. What is the angle between  $\vec{a}$  and  $\vec{b}$ , if  $\vec{a} \cdot \vec{b} = 3$  and  $|\vec{a} \times \vec{b}| = 3\sqrt{3}$ .

### TWO MARK QUESTIONS

- Q.1. A vector  $\vec{r}$  is inclined to x – axis at  $45^\circ$  and y-axis at  $60^\circ$  if  $|\vec{r}| = 8$  units. find  $\vec{r}$ .
- Q.2. if  $|\vec{a} + \vec{b}| = 60$ ,  $|\vec{a} - \vec{b}| = 40$  and  $|\vec{b}| = 46$  find  $|\vec{a}|$

Q.3. Write the projection of  $\vec{b} + \vec{c}$  on  $\vec{a}$  where

$$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k} \text{ and } \vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$$

Q.4. If the points  $(-1, -1, 2)$ ,  $(2, m, 5)$  and  $(3, 11, 6)$  are collinear, find the value of  $m$ .

Q.5. For any three vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$  write value of the following.

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$$

Q.6. If  $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = 144$  and  $|\vec{a}| = 4$ . Find the value of  $|\vec{b}|$ .

Q.7. If for any two vectors  $\vec{a}$  and  $\vec{b}$ ,

$$(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = \lambda [(\vec{a})^2 + (\vec{b})^2] \text{ then write the value of } \lambda.$$

Q.8. if  $\vec{a}, \vec{b}$  are two vectors such that  $|(\vec{a} + \vec{b})| = |\vec{a}|$  then prove that  $2\vec{a} + \vec{b}$  is perpendicular to  $\vec{b}$ .

Q.9. Show that vectors  $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}, \vec{c} = 2\hat{i} + \hat{j} - 4\hat{k} \text{ form a right angle triangle.}$$

Q.10. If  $\vec{a}, \vec{b}, \vec{c}$  are three vectors such that  $\vec{a} + \vec{b} + \vec{c} = 0$  and  $|\vec{a}| = 5$ ,  $|\vec{b}| = 12$ ,  $|\vec{c}| = 13$ , then find  $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$

Q.11. The two vectors  $\hat{i} + \hat{j}$  and  $3\hat{i} - \hat{j} + 4\hat{k}$  represents the two sides AB and AC respectively of  $\Delta ABC$ , find the length of median through A.

## FOUR MARK QUESTIONS

1. The points A, B and C with position vectors  $3\hat{i} - y\hat{j} + 2\hat{k}$ ,  $5\hat{i} - \hat{j} + \hat{k}$  and  $3x\hat{i} + 3\hat{j} - \hat{k}$  are collinear. Find the values of x and y and also the ratio in which the point B divides AC.
2. If sum of two unit vectors is a unit vector, prove that the magnitude of their difference is  $\sqrt{3}$ .
3. Let  $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$ ,  $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$ . Find a vector  $\vec{d}$  which is perpendicular to both  $\vec{a}$  and  $\vec{b}$  and satisfying  $\vec{d} \cdot \vec{c} = 21$
4. If  $\hat{a}$  and  $\hat{b}$  are unit vectors inclined at an angle  $\theta$  then proved that
  - (i)  $\cos \frac{\theta}{2} = \frac{1}{2} |\hat{a} + \hat{b}|$
  - (ii)  $\tan \frac{\theta}{2} = \left| \frac{\hat{a} - \hat{b}}{\hat{a} + \hat{b}} \right|$
5. If  $\vec{a}, \vec{b}, \vec{c}$  are three mutually perpendicular vectors of equal magnitude. Prove that  $\vec{a} + \vec{b} + \vec{c}$  is equally inclined with vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$ . Also find angles.
6. For any vector  $\vec{a}$  prove that  $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$
7. Show that  $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$
8. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are the position vectors of vertices A, B, C of a  $\Delta ABC$ , show that the area of triangle ABC is  $\frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$ . Deduce the condition for points  $\vec{a}, \vec{b}$  and  $\vec{c}$  to be collinear.

9. Let  $\vec{a}, \vec{b}$  and  $\vec{c}$  be unit vectors such that  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$  and the angle between  $\vec{b}$  and  $\vec{c}$  is  $\pi/6$ , prove that  $\vec{a} = \pm 2(\vec{b} \times \vec{c})$ .
10. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are three vectors such that  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ , then prove that  $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ .
11. If  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{c} = \hat{j} - \hat{k}$  are given vectors, then find a vector  $\vec{b}$  satisfying the equations  $\vec{a} \times \vec{b} = \vec{c}$  and  $\vec{a} \cdot \vec{b} = 3$ .
12. Let  $\vec{a}, \vec{b}$  and  $\vec{c}$  be three non zero vectors such that  $\vec{c}$  is a unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$ . If the angle between  $\vec{a}$  and  $\vec{b}$  is  $\pi/6$ , prove that  $[\vec{a}\vec{b}\vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$ .
13. If the vectors  $\vec{\alpha} = a\hat{i} + \hat{j} + \hat{k}$ ,  $\vec{\beta} = \hat{i} + b\hat{j} + \hat{k}$  and  $\vec{\gamma} = \hat{i} + \hat{j} + c\hat{k}$  are coplanar, then prove that  $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$  where  $a \neq 1, b \neq 1$  and  $c \neq 1$ .
14. Find the altitude of a parallelepiped determined by the vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$  if the base is taken as parallelogram determined by  $\vec{a}$  and  $\vec{b}$  and if  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{b} = 2\hat{i} + 4\hat{j} - \hat{k}$  and  $\vec{c} = \hat{i} + \hat{j} + 3\hat{k}$ .
15. Prove that the four points  $(4\hat{i} + 5\hat{j} + \hat{k})$ ,  $-(\hat{j} + \hat{k})$ ,  $(3\hat{i} + 9\hat{j} + 4\hat{k})$  and  $4(-\hat{i} + \hat{j} + \hat{k})$  are coplanar.
16. If  $|\vec{a}| = 3$ ,  $|\vec{b}| = 4$  and  $|\vec{c}| = 5$  such that each is perpendicular to sum of the other two, find  $|\vec{a} + \vec{b} + \vec{c}|$ .
17. Decompose the vector  $6\hat{i} - 3\hat{j} - 6\hat{k}$  into vectors which are parallel and perpendicular to the vector  $\hat{i} + \hat{j} + \hat{k}$ .

18. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are vectors such that  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ ,  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ ,  $a \neq 0$ , then show that  $\vec{b} = \vec{c}$ .
19. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are three non zero vectors such that  $\vec{a} \times \vec{b} = \vec{c}$  and  $\vec{b} \times \vec{c} = \vec{a}$ . Prove that  $\vec{a}, \vec{b}$  and  $\vec{c}$  are mutually at right angles and  $|\vec{b}| = 1$  and  $|\vec{c}| = |\vec{a}|$ .
20. Simplify  $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}]$
21. If  $[\vec{a}\vec{b}\vec{c}] = 2$ , find the volume of the parallelepiped whose co-terminus edges are  $2\vec{a} + \vec{b}$ ,  $2\vec{b} + \vec{c}$ ,  $2\vec{c} + \vec{a}$ .
22. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are three vectors such that  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$  and  $|\vec{a}| = 3$ ,  $|\vec{b}| = 5$ ,  $|\vec{c}| = 7$ , find the angle between  $\vec{a}$  and  $\vec{b}$ .
23. The magnitude of the vector product of the vector  $\hat{i} + \hat{j} + \hat{k}$  with a unit vector along the sum of the vector  $2\hat{i} + 4\hat{j} - 5\hat{k}$  and  $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$  is equal to  $\sqrt{2}$ . Find the value of  $\lambda$ .
24. If  $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$  and  $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ , prove that  $(\vec{a} - \vec{d})$  is parallel to  $(\vec{b} - \vec{c})$ , where  $\vec{a} \neq \vec{d}$  and  $\vec{b} \neq \vec{c}$ .
25. Find a vector of magnitude  $\sqrt{171}$  which is perpendicular to both of the vectors  $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$  and  $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ .
26. If  $a$  is a nonzero real number, prove that the vectors  $\vec{\alpha} = a\hat{i} + 2a\hat{j} - 3a\hat{k}$ ,  $\vec{\beta} = (2a + 1)\hat{i} + (2a + 3)\hat{j} + (a + 1)\hat{k}$  and  $\vec{\gamma} = (3a + 5)\hat{i} + (a + 5)\hat{j} + (a + 2)\hat{k}$  are never coplanar.



27. If  $\vec{\alpha} = 3\hat{i} - \hat{j}$  and  $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$  then express  $\vec{\beta}$  in the form of  $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$ , where  $\vec{\beta}_1$  is parallel to  $\vec{\alpha}$  and  $\vec{\beta}_2$  is perpendicular to  $\vec{\alpha}$ .
28. Find a unit vector perpendicular to plane ABC, when position vectors of A,B,C are  $3\hat{i} - \hat{j} + 2\hat{k}$ ,  $\hat{i} - \hat{j} - 3\hat{k}$  and  $4\hat{i} - 3\hat{j} + \hat{k}$  respectively.
29. Find a unit vector in XY plane which makes an angle  $45^\circ$  with the vector  $\hat{i} + \hat{j}$  at angle of  $60^\circ$  with the vector  $3\hat{i} - 4\hat{j}$ .
30. Suppose  $\vec{a} = \lambda\hat{i} - 7\hat{j} + 3\hat{k}$ ,  $\vec{b} = \lambda\hat{i} + \hat{j} + 2\lambda\hat{k}$ . If the angle between  $\vec{a}$  and  $\vec{b}$  is greater than  $90^\circ$ , then prove that  $\lambda$  satisfies the inequality  $-7 < \lambda < 1$ .
31. Let  $\vec{v} = 2\hat{i} + \hat{j} - \hat{k}$  and  $\vec{w} = \hat{i} + 3\hat{k}$ . If  $u$  is a unit vector, then find the maximum value of the scalar triple products  $\vec{u}, \vec{v}, \vec{w}$ .
32. If  $\vec{a} = \hat{i} - \hat{k}$ ,  $\vec{b} = x\hat{i} + \hat{j} + (1-x)\hat{k}$  and  $\vec{c} = y\hat{i} + x\hat{j} + (1+x-y)\hat{k}$  then prove that  $[\vec{a}\vec{b}\vec{c}]$  depends upon neither  $x$  nor  $y$ .
33. A, b and c are distinct non negative numbers, if the vectors  $a\hat{i} + a\hat{j} + c\hat{k}$ ,  $\hat{i} + \hat{k}$  and  $c\hat{i} + c\hat{j} + b\hat{k}$  lie in a plane, then prove that c is the geometric mean of a and b.
34. If  $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$  and vectors  $(1, a, a^2)$ ,  $(1, b, b^2)$  and  $(1, c, c^2)$  are non-coplanar, then find the value of abc.

## Answers

### ONE MARK QUESTIONS

1.  $(7, 3, 0)$
2.  $x = -1, y = 2$
3.  $\frac{1}{\sqrt{21}}(4\hat{i} + 2\hat{j} - \hat{k})$
4.  $\sqrt{\frac{5}{2}}(3\hat{i} + \hat{j})$
5.  $\lambda = \frac{16}{5}$
6.  $\frac{2}{3}$
7.  $\vec{a}$  and  $\vec{b}$  are perpendicular
8.  $\frac{27}{2}$
9. 0
10. 4
11. -5
12. ....
13.  $-2\hat{i} + 4\hat{j} + 4\hat{k}$
14. Either  $\vec{a} = 0$  or  $\vec{b} = 0$  or  $\vec{a} \perp \vec{b}$
15.  $45^\circ$
16.  $5\sqrt{3}$  sq. Units
17. 1

$$18. \quad -\vec{a} + 4\vec{b}$$

$$19. \quad \lambda = 5$$

$$20. \quad a = 3$$

$$21. \quad \lambda = 1$$

$$22. \quad \left(5, \frac{14}{3}, -6\right)$$

$$23. \quad \frac{\pi}{3}$$

$$24. \quad \text{(i) Hint and sol.} \\ x + y + z = 3 - 2 - 1 = 0$$

$$25. \quad 3\hat{i} - 6\hat{j} + 6\hat{k}$$

$$26. \quad \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$$

$$27. \quad p = -\frac{1}{3}$$

$$28. \quad \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$$

## TWO MARK QUESTIONS

$$1. \quad 4(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$$

$$2. \quad 22$$

$$3. \quad 2$$

$$4. \quad m = 8$$

$$5. \quad 0$$

$$6. \quad 3$$

$$7. \quad \lambda = 2$$

$$8. \quad \text{—}$$

$$9. \quad \text{—}$$

$$10. \quad -169$$

$$11. \quad 2\sqrt{2}$$

$$12. \quad \text{Hint we have } \overrightarrow{PQ} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR} \\ \Rightarrow \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{OR} - \overrightarrow{OQ} \\ \Rightarrow \overrightarrow{PQ} = \overrightarrow{QR}$$

As Q is common point for both vectors Hence P, Q, R are collinear.

$$13. \quad 13\sqrt{3}$$

$$14. \quad \boxed{\lambda = -4}$$

$$15. \quad \text{Hint and sol}$$

$$\text{given } |\vec{a}| = 2, |\vec{b}| = 3, \text{ and } \vec{a} \cdot \vec{b} = 4$$

$$\text{Now consider } |\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$|\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \quad \because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$(\vec{a} - \vec{b})^2 = (2)^2 - 2(4) + (3)^2 = 5$$

$$|\vec{a} - \vec{b}| = \sqrt{5}$$

### FOUR MARK QUESTIONS

1.  $x = 3, y = 3, 1:2$

3.  $\vec{d} = 7\hat{i} - 7\hat{j} - 7\hat{k}$

5.  $\cos^{-1} \frac{1}{\sqrt{3}}$

11.  $\vec{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

14.  $\frac{4}{\sqrt{38}}$  units

16.  $5\sqrt{2}$

17.  $(-\hat{i} - \hat{j} - \hat{k}) + (7\hat{i} - 2\hat{j} - 5\hat{k})$

20. 0

21. 18 cu. Units

22.  $60^\circ$

23.  $\lambda = 1$

25.  $\hat{i} - 11\hat{j} - 7\hat{k}$

27.  $\vec{\beta} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} + 3\hat{k}\right)$

28.  $\frac{-1}{\sqrt{165}}(10\hat{i} + 7\hat{j} - 4\hat{k})$

29.  $\frac{13}{\sqrt{170}}\hat{i} + \frac{1}{\sqrt{170}}\hat{j}$

31.  $\sqrt{59}$

34. -1

## CHAPTER–11

### THREE-DIMENSIONAL GEOMETRY

#### POINTS TO REMEMBER

- **Distance Formula:** Distance (d) between two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section Formula:** line segment AB is divided by P (x, y, z) in ratio m:n

(a) Internally	(b) Externally
$\left( \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$	$\left( \frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right)$

- **Direction ratio** of a line through  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are  $x_2 - x_1, y_2 - y_1, z_2 - z_1$

- **Direction cosines** of a line having direction ratios as a, b, c are:

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of line in space:**

Vector form	Cartesian form
(i) Passing through point $\vec{a}$ and parallel to vector $\vec{b}$ ; $\vec{r} = \vec{a} + \lambda \vec{b}$	(i) Passing through point $(x_1, y_1, z_1)$ and having direction ratios a, b, c;

	$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
(ii) Passing through two points $\vec{a}$ and $\vec{b}$ ; $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$	(ii) Passing through two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ ; $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

- **Angle between two lines:**

Vector form	Cartesian form
(i) For lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ $\cos \theta = \frac{ \vec{b}_1 \cdot \vec{b}_2 }{ \vec{b}_1   \vec{b}_2 }$	(ii) For lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ $\cos \theta = \frac{ a_1 a_2 + b_1 b_2 + c_1 c_2 }{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$
(iii) Lines are perpendicular if $\vec{b}_1 \cdot \vec{b}_2 = 0$	(ii) Lines are perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$
(iv) Lines are parallel if $\vec{b}_1 = k \vec{b}_2$ ; $k \neq 0$	(i) Lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- **Equation of plane:**

If p is length of perpendicular from origin to plane and $\hat{n}$ is unit vector normal to plane $\vec{r} \cdot \hat{n} = p$	If p is length of perpendicular from origin to plane and l, m, n are d.c.s of normal to plane $lx + my + nz = p$
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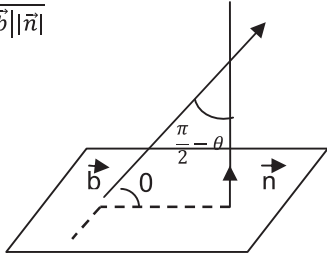
Passing through $\vec{a}$ and $\vec{n}$ is normal to plane : $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$	Passing through $(x_1, y_1, z_1)$ and a, b, c are d.r.s of normal to plane:  $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
Passing through three non collinear points $\vec{a}, \vec{b}, \vec{c}$ :  $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$	Passing through three non collinear points $(x_1, y_1, z_1)(x_2, y_2, z_2)(x_3, y_3, z_3)$ :  $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
If a, b, c are intercepts on co-ordinate axes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$	If $x_1, y_1, z_1$ are intercepts on coordinate axes $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$
Plane passing through line of intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is  $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$ ( $\lambda$ =real no.)	Plane passing through the line of intersection of planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is  $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

• **Angle between planes:**

Angle $\theta$ between planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is  $\cos \theta = \frac{ \vec{n}_1 \cdot \vec{n}_2 }{ \vec{n}_1  \vec{n}_2 }$	Angle $\theta$ between planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is  $\cos \theta = \frac{ a_1a_2 + b_1b_2 + c_1c_2 }{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$
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Planes are perpendicular iff $\vec{n}_1 \cdot \vec{n}_2 = 0$	Planes are perpendicular iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$
Planes are parallel iff $\vec{n}_1 = \lambda \vec{n}_2 ; \lambda \neq 0$	Planes are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

• **Angle between line and plane:**

<p>Angle <math>\theta</math> between line <math>\vec{r} = \vec{a} + \lambda \vec{b}</math> and plane <math>\vec{r} \cdot \vec{n} = d</math> is <math>\sin \theta = \cos(90^\circ - \theta)</math></p> <p><math display="block">= \frac{\vec{b} \cdot \vec{n}}{ \vec{b}   \vec{n} }</math></p> 	<p>Angle <math>\theta</math> between line <math>\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}</math> and plane <math>a_2x + b_2y + c_2z = d</math> is</p> <p><math display="block">\sin \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}</math></p>
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• **Distance of a point from a plane**

<p>The perpendicular distance <math>p</math> from the point <math>P</math> with position vector <math>\vec{a}</math> to the plane <math>\vec{r} \cdot \vec{n} = d</math> is given by</p> <p><math display="block">p = \frac{ \vec{a} \cdot \vec{n} - d }{ \vec{n} }</math></p>	<p>The perpendicular distance <math>p</math> from the point <math>P (x_1, y_1, z_1)</math> to the plane <math>Ax + By + Cz + D = 0</math> is given by</p> <p><math display="block">p = \frac{ Ax_1 + By_1 + Cz_1 + D }{\sqrt{A^2 + B^2 + C^2}}</math></p>
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- Coplanarity**

<p>Two lines <math>\vec{r} = \vec{a}_1 + \lambda \vec{b}_1</math> and <math>\vec{r} = \vec{a}_2 + \mu \vec{b}_2</math> are coplanar iff</p> $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$	<p>Two lines <math>\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}</math> and <math>\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}</math> are coplanar iff</p> $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
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- Shortest distance between two skew lines**

<p>The shortest distance between lines <math>\vec{r} = \vec{a}_1 + \lambda \vec{b}_1</math> and <math>\vec{r} = \vec{a}_2 + \mu \vec{b}_2</math> is</p> $d = \left  \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{ \vec{b}_1 \times \vec{b}_2 } \right $	<p>The shortest distance between <math>\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}</math> and <math>\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}</math> is</p> $d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}}$ <p>Where</p> $D = \{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2\}$
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## ONE MARK QUESTIONS

1. What is the distance of point (a, b, c) from x-axis?
2. What is the angle between the lines  $2x = 3y = -z$  and  $6x = -y = -4z$ ?
3. Write the equation of a line passing through (2, -3, 5) and parallel to line  $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z+1}{-1}$ .
4. Write the equation of a line through (1, 2, 3) and parallel to  $\vec{r} \cdot (\hat{i} - \hat{j} + 3\hat{k}) = 5$ .
5. What is the value of  $\lambda$  for which the lines  $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{\lambda}$  and  $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$  are perpendicular to each other?
6. Write line  $\vec{r} = (\hat{i} - \hat{j}) + \lambda (2\hat{j} - \hat{k})$  into Cartesian form.
7. If the direction ratios of a line are 1, -2, 2 then what are the direction cosines of the line?
8. Find the angle between the planes  $2x - 3y + 6z = 9$  and  $xy$  - plane.
9. Write equation of a line passing through (0, 1, 2) and equally inclined to co-ordinate axes.
10. What is the perpendicular distance of plane  $2x - y + 3z = 10$  from origin?

11. What is the y-intercept of the plane  $x - 5y + 7z = 10$ ?
12. What is the distance between the planes  $2x + 2y - z + 2 = 0$  and  $4x + 4y - 2z + 5 = 0$ .
13. What is the equation of the plane which cuts off equal intercepts of unit length on the coordinate axes?
14. Are the planes  $x + y - 2z + 4 = 0$  and  $3x + 3y - 6z + 5 = 0$  intersecting?
15. What is the equation of the plane through the point (1, 4, -2) and parallel to the plane  $-2x + y - 3z = 7$ ?
16. Write the vector equation of the plane which is at a distance of 8 units from the origin and is normal to the vector  $(2\hat{i} + \hat{j} + 2\hat{k})$ .
17. What is equation of the plane if the foot of perpendicular from origin to this plane is (2, 3, 4)?
18. Find the angles between the planes  $\vec{r} \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) = 1$  and  $\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = 0$ .
19. If O is origin  $OP = 3$  with direction ratios proportional to -1, 2, -2 then what are the coordinates of P?
20. What is the distance between the line  $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda (\hat{i} + \hat{j} + 4\hat{k})$  from the plane  $\vec{r} \cdot (-\hat{i} + 5\hat{j} - \hat{k}) + 5 = 0$ .
21. Write the line  $2x = 3y = 4z$  in vector form.

22. The line  $\frac{x-4}{1} = \frac{2y-4}{2} = \frac{k-z}{-2}$  lies exactly in the plane  $2x - 4y + z = 7$ .  
Find the value of k.

## TWO MARK QUESTIONS

1. What is the angle between the line  $\frac{x+1}{3} = \frac{2y-1}{4} = \frac{2-z}{-4}$  and the plane  $2x + y - 2z + 4 = 0$
2. Find the equation of a line passing through  $(2, 0, 5)$  and which is parallel to line  $6x - 2 = 3y + 1 = 2z - 2$
3. Find the equation of the plane passing through the points  $(2, 3, -4)$  and  $(1, -1, 3)$  and parallel to the x – axis.
4. Find the distance between the planes  $2x + 3y - 4z + 5 = 0$  and  $\hat{r} \cdot (4\hat{i} + 6\hat{j} - 8\hat{k}) = 11$
5. The equation of a line are  $5x - 3 = 15y + 7 = 3 - 10z$ . Write the direction cosines of the line
6. If a line makes angle  $\alpha, \beta, \gamma$  with Co-ordinate axis then what is the value of  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$
7. Find the equation of a line passing through the point  $(2, 0, 1)$  and parallel to the line whose equation is  $\vec{r} = (2\lambda + 3)\hat{i} + (7\lambda - 1)\hat{j} + (-3\lambda + 2)\hat{k}$
8. The plane  $2x - 3y + 6z - 11 = 0$  makes an angle  $\sin^{-1} \alpha$  with x – axis. Find the value of  $\alpha$ .
9. If  $4x + 4y - cz = 0$  is the equation of the plane passing through the origin that contains the line  $\frac{x+5}{2} = \frac{y}{3} = \frac{z-7}{4}$ , then find the value of c.

10. Find the equation of the plane passing through the point  $(-2, 1, -3)$  and making equal intercept on the coordinate axes.
11. Write the sum of intercepts cut off by the plane  $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$  on the three axis.

### FOUR MARK QUESTIONS

1. Find the equation of a plane containing the points  $(0, -1, -1)$ ,  $(-4, 4, 4)$  and  $(4, 5, 1)$ . Also show that  $(3, 9, 4)$  lies on that plane.
2. Find the equation of the plane which is perpendicular to the plane  $\vec{r} \cdot (5\hat{i} + 3\hat{j} + 6\hat{k}) + 8 = 0$  and which is containing the line of intersection of the planes  $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 4$  and  $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ .
3. Find the distance of the point  $(3, 4, 5)$  from the plane  $x + y + z = 2$  measured parallel to the line  $2x = y = z$ .
4. Find the equation of the plane passing through the intersection of two planes  $x + 2y + 3z - 5 = 0$  and  $3x - 2y - z + 1 = 0$  and cutting equal intercepts on x-axis and z-axis.
5. Find vector and Cartesian equation of a line passing through a point with position vector  $2\hat{i} - \hat{j} + \hat{k}$  and which is parallel to the line joining the points with position vectors  $-\hat{i} + 4\hat{j} + \hat{k}$  and  $\hat{i} + 2\hat{j} + 2\hat{k}$ .
6. Find the equation of the plane passing through the point  $(3, 4, 2)$  and  $(7, 0, 6)$  and is perpendicular to the plane  $2x - 5y = 15$ .

7. Find equation of plane through line of intersection of planes  $\vec{r} \cdot (2\hat{i} + 6\hat{j}) + 12 = 0$  and  $\vec{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 0$  which is at a unit distance from origin.
8. Find the image of point  $(3, -2, 1)$  in the plane  $3x - y + 4z = 2$ .
9. Find image (reflection) of the point  $(7, 4, -3)$  in the line  $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ .
10. Find equation of a plane passing through the points  $(2, -1, 0)$  and  $(3, -4, 5)$  and parallel to the line  $2x = 3y = 4z$ .
11. Find the distance of the point  $(-1, -5, -10)$  from the point of intersection of line  $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$  and the plane  $x - y + z = 5$ .
12. Find the distance of the point  $(1, -2, 3)$  from the plane  $x - y + z = 5$ , measured parallel to the line  $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ .
13. Find the equation of the plane passing through the intersection of two plane  $3x - 4y + 5z = 10$ ,  $2x + 2y - 3z = 4$  and parallel to the line  $x = 2y = 3z$ .
14. Find the equation of the planes parallel to the plane  $x - 2y + 2z - 3 = 0$  whose perpendicular distance from the point  $(1, 2, 3)$  is 1 unit.
15. Show that the lines  $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$  and  $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$  intersect each other. Find the point of intersection.

16. Find the shortest distance between the lines:

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and}$$

$$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 5\hat{k}).$$

17. Find the distance of the point  $(-2, 3, -4)$  from the line  $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$  measured parallel to the plane  $4x + 12y - 3z + 1 = 0$ .

18. Find the equation of plane passing through the point  $(-1, -1, 2)$  and perpendicular to each of the plane

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} - 3\hat{k}) = 2 \text{ and } \vec{r} \cdot (5\hat{i} - 4\hat{j} + \hat{k}) = 6$$

19. Find the equation of a plane passing through  $(-1, 3, 2)$  and parallel to each of the line  $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$  and  $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$

20. Show that the plane  $\vec{r} \cdot (\hat{i} - 3\hat{j} + 5\hat{k}) = 7$  contains the line

$$\vec{r} = (\hat{i} + 3\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + \hat{j}).$$

### SIX MARK QUESTIONS

1. Check the co planarity of lines

$$\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda(-3\hat{i} + \hat{j} + 5\hat{k}).$$

$$\vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \mu(-\hat{i} + 2\hat{j} + 5\hat{k})$$

If they are coplanar, find equation of the plane containing the lines.

2. Find shortest distance between the lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{5-y}{2} = \frac{z-7}{1}$$

3. Find the shortest distance between the lines:

$$\vec{r} = (1 - \lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$$

$$\vec{r} = (\mu + 1)\hat{i} + (2\mu - 1)\hat{j} - (2\mu + 1)\hat{k}$$

4. A variable plane is at a constant distance 3 p from the origin and meets the coordinates axes in A, B and C. If the centroid of  $\Delta ABC$  is  $(\alpha, \beta, \gamma)$ , then show that  $\alpha^{-2} + \beta^{-2} + \gamma^{-2} = p^{-2}$

5. A vector  $\vec{n}$  of magnitude 8 units is inclined to x-axis at  $45^\circ$ , y axis at  $60^\circ$  and an acute angle with z-axis. If a plane passes through a point  $(\sqrt{2}, -1, 1)$  and is normal to  $\vec{n}$ , find its equation in vector form.



6. Find the foot of perpendicular from the point  $2\hat{i} - \hat{j} + 5\hat{k}$  on the line  $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$ . Also find the length of the perpendicular.
7. A line makes angles  $\alpha, \beta, \gamma, \delta$  with the four diagonal of a cube. Prove that  $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$
8. Find the equation of the plane passing through the intersection of planes  $2x + 3y - z = -1$  and  $x + y - 2z + 3 = 0$  and perpendicular to the plane  $3x - y - 2z = 4$ . Also find the inclination of this plane with xy-plane.
9. Find the length and the equations of the line of shortest distance between the lines  $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$  and  $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$ .
10. Show that  $\frac{x-1}{2} = \frac{y+1}{3} = z$  and  $\frac{x+1}{5} = \frac{y-2}{2}, z = 2$ . do not intersect each other.

## ANSWERS

### ONE MARK QUESTIONS

- |  |  |
|--|--|
| 1. $\sqrt{b^2 + c^2}$  | 13. $x + y + z = 1$                                      |
| 2. $90^\circ$  | 14. No   |
| 3. $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$                                    | 15. $-2x + y - 3z = 8$                                   |
| 4. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 3\hat{k})$ | 16. $\vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24$ |
|  | 17. $2x + 3y + 4z = 29$                                  |

5.  $\lambda = 2$

6.  $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$

7.  $\pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{2}{3}$

8.  $\cos^{-1}(6/7)$

9.  $\frac{x}{a} = \frac{y-1}{a} = \frac{z-2}{a},$

$a \in R - \{0\}$

10.  $\frac{10}{\sqrt{14}}$

11.  $-2$

12.  $\frac{1}{6}$

18.  $\cos^{-1}\left(\frac{11}{21}\right)$

19.  $(-1, 2, -2)$

20.  $\frac{10}{3\sqrt{3}}$

21.  $\vec{r} = \vec{0} + \lambda(6\hat{i} + 4\hat{j} + 3\hat{k}).$

22.  $k = 7$

## TWO MARK QUESTIONS

Q.1  $0^\circ$  (line is parallel to plane )

Q.8  $\alpha = \frac{2}{7}$

Q.2  $\frac{x-2}{1} = \frac{y}{2} = \frac{z-5}{3}$

Q.9  $C = 5$

Q.3  $7y + 4z = 5$

Q.10  $x + y + z = -4$

Q.4  $\frac{21}{2\sqrt{29}}$  units

Q.11  $\frac{5}{2}$

Q.5  $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$

Q.6  $2$

Q.7  $\vec{r} = (2\hat{i} + \hat{k}) + \lambda(2\hat{i} + 7\hat{j} - \hat{k})$

## FOUR MARK QUESTIONS

1.  $5x - 7y + 11z + 4 = 0$
2.  $\vec{r} \cdot (-51\hat{i} - 15\hat{j} + 50\hat{k}) = 173$
3. 6 units
4.  $5x + 2y + 5z - 9 = 0$
5.  $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$   
and  $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$
6.  $5x + 2y - 3z - 17 = 0$
7.  $\vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) + 3 = 0$  or  $\vec{r} \cdot (-\hat{i} + 2\hat{j} - 2\hat{k}) + 3 = 0$
8. (0, -1, -3)
9.  $\left(-\frac{51}{7}, -\frac{18}{7}, \frac{43}{7}\right)$
10.  $29x - 27y - 22z = 85$
11. 13
12. 1 unit
13.  $x - 20y + 27z = 14$
14.  $x - 2y + 2z = 0$  and  $x - 2y + 2z = 6$
15.  $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$
16.  $\frac{1}{\sqrt{6}}$
17.  $\frac{17}{2}$  units
18.  $\vec{r} \cdot (9\hat{i} + 17\hat{j} + 23\hat{k}) = 20$
19.  $2x - 7y + 4z + 15 = 0$

## SIX MARK QUESTIONS

1.  $x - 2y + z = 0$
2.  $2\sqrt{29}$  units
3.  $\frac{8}{\sqrt{29}}$
5.  $\vec{r} \cdot (\sqrt{2}\hat{i} + \hat{j} + \hat{k}) = 2$
6.  $(1, 2, 3), \sqrt{14}$
8.  $7x + 13y + 4z =$   
 $9, \cos^{-1}\left(\frac{4}{\sqrt{234}}\right)$
9.  $SD = 14$  units,  
 $\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

## CHAPTER 12

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### LINEAR PROGRAMMING

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#### POINTS TO REMEMBER

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- Linear programming is the process used to obtain minimum or maximum value of the linear objectives function under known linear constraints.
- **Objective Functions:** Linear function  $z = ax + by$  where  $a$  and  $b$  are constants, which has to be maximized or minimized is called a linear objective function.
- **Constraints:** the linear inequalities or inequations or restrictions on the variables of a linear programming problem.
- **Feasible Region:** It is defined as a set of points which satisfy all the constraints.
- **To Find Feasible Region:** Draw the graph of all the linear in equations and shade common region determined by all the constraints.
- **Feasible Solutions:** Points within and on the boundary of the feasible region represents feasible solutions of the constraints.
- **Optimal Feasible Solution:** Feasible solution which optimizes the objective function is called optimal feasible solution.

#### Long Answer Type Questions (6 Marks)

1. Solve the following L.P.P. graphically

Minimise and maximise

$$z = 3x + 9y$$

Subject to the constraints

$$x + 3y \leq 60$$

$$x + y \geq 10$$

$$x \leq y$$

$$x \geq 0, y \geq 0$$

2. Determine graphically the minimum value of the objective function  $z = -50x + 20y$ , subject to the constraints.

$$2x - y \geq -5$$

$$3x + y \geq 3$$

$$2x - 3y \leq 12$$

$$x \geq 0, \quad y \geq 0$$

3. Two tailors A and B earn Rs. 150 and Rs. 200 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce atleast 60 shirts and 32 pants at a minimum labour cost? Solve the problem graphically.
4. There are two types of fertilisers A and B. A consists of 10% nitrogen and 6% phosphoric acid and B consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If A costs Rs. 6 per kg and B costs Rs. 5 per kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at minimum cost. What is the minimum cost?
5. A man has Rs. 1500 to purchase two types of shares of two different companies  $S_1$  and  $S_2$ . Market price of one share of  $S_1$  is Rs. 180 and  $S_2$  is Rs 120. He wishes to purchase a maximum of ten shares only. If

one share of type  $S_1$  gives a yield of Rs. 11 and of type  $S_2$  yields Rs. 8 then how much shares of each type must be purchased to get maximum profit? And what will be the maximum profit?

6. A company manufactures two types of lamps say A and B. Both lamps go through a cutter and then a finisher. Lamp A requires 2 hours of the cutter's time and 1 hours of the finisher's time. Lamp B requires 1 hour of cutter's and 2 hours of finisher's time. The cutter has 100 hours and finisher has 80 hours of time available each month. Profit on one lamp A is Rs. 7.00 and on one lamp B is Rs. 13.00. Assuming that he can sell all that he produces, how many of each type of lamps should be manufactured to obtain maximum profit?
7. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for atmost 20 items. A fan and sewing machine cost Rs. 360 and Rs. 240 respectively. He can sell a fan at a profit of Rs. 22 and sewing machine at a profit of Rs. 18. Assuming that he can sell whatever he buys, how should he invest his money to maximise his profit?
8. If a young man rides his motorcycle at 25 km/h, he has to spend Rs. 2 per km on petrol. If he rides at a faster speed of 40 km/h, the petrol cost increase to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to cover the maximum distance within one hour. Express this as L.P.P. and then solve it graphically.
9. A producer has 20 and 10 units of labour and capital respectively which he can use to produce two kinds of goods X and Y. To produce one unit of X, 2 units of capital and 1 unit of labour is required. To produce one unit of Y, 3 units of labour and 1 unit of capital is required. If X and

Y are priced at Rs. 80 and Rs. 100 per unit respectively, how should the producer use his resources to maximise the total revenue?

10. A factory owner purchases two types of machines A and B for his factory. The requirements and limitations for the machines are as follows:

Machine	Area Occupied	Labour Force	Daily Output(In units)
<b>A</b>	1000 $m^2$	12 men	50
<b>B</b>	1200 $m^2$	8 men	40

He has maximum area of 7600  $m^2$  available and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output?

11. A manufacturer makes two types of cups A and B. Three machines are required to manufacture the cups and the time in minutes required by each in as given below:

Types of Cup	Machines		
	I	II	III
<b>A</b>	12	18	6
<b>B</b>	6	0	9

Each machine is available for a maximum period of 6 hours per day. If the profit on each cup A is 75 paisa and on B is 50 paisa, find how many cups of each type should be manufactures to maximise the profit per day.

12. A company produces two types of belts A and B. Profits on these belts are Rs. 2 and Rs. 1.50 per belt respectively. A belt of type A requires twice as much time as belt of type B. The company can produce at most 1000 belts of type B per day. Material for 800 belts per day is

available. At most 400 buckles for belts of type A and 700 for type B are available per day. How much belts of each type should the company produce so as to maximize the profit?

13. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each second class ticket. The airline reserves at least 20 seats for first class. However at least four times as many passengers prefer to travel by second class than by first class. Determine how many tickets of each type must be sold to maximize profit for the airline.
14. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of Rs. 5 and Rs. 4 per unit respectively. One unit of food A contains 200 units of vitamins, 1 unit of minerals and 40 units of calories whereas one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 units of calories. Find what combination of the food A and B should be used to have least cost but it must satisfy the requirements of the sick person.
15. Anil wants to invest at most Rs, 12000 in bonds A and B. According to the rules, he has to invest at least Rs. 2000 in Bond A and at least Rs. 4000 in bond B. If the rate of interest on bond A and B are 8% and 10% per annum respectively, how should he invest this money for maximum interest? Formulate the problem as L.P.P. and solve graphically.



## LINEAR PROGRAMMING

### One Mark Questions

Chose the Correct option for the following MCQ's(Single option is correct).

1. Objective Function of a L.P.P. is
  - (a) A constraint
  - (b) A function to be optimised
  - (c) A relation between the variables
  - (d) None of these
2. The optimal value of the objective function is attained at the points:
  - (a) Given by intersections of equations with axis only
  - (b) Given by intersections of inequations with x-axis only
  - (c) Given by corner points of the feasible region
  - (d) None of these
3. The solution set of the inequation  $2x + y > 5$  is
  - (a) Open half-plane that contains the origin
  - (b) Open half-plane not containing the origin
  - (c) Whole xy-plane except the points lying on the line  $2x + y = 5$
  - (d) None of these
4. If the constraints in a liner programming problem are changed, then
  - (a) The problem is to be re-evaluled
  - (b) Solution not defined
  - (c) The objective function has to be modified
  - (d) The change in constraints is ignored

5. Which of the following statements is correct?
- (a) Every L.P.P. admits an optimal solution
  - (b) A L.P.P. admits unique optimal solution
  - (c) If a L.P.P. admits two optimal solutions it has an infinite number of optimal solutions
  - (d) None of these
6. Solution set of inequation  $x \geq 0$  is
- (a) Half-plane on the left of y-axis.
  - (b) Half-plane on the right of y axis excluding the points on y-axis.
  - (c) Half-plane on the right of y-axis including the points on y-axis.
  - (d) None of these
7. Solution set of the inequation  $y \leq 0$  is
- (a) Half-plane below the x-axis excluding the points on x-axis
  - (b) Half-plane below the x-axis including the point on x-axis.
  - (c) Half-plane above the x-axis.
  - (d) None of these
8. Regions represented by equations  $x \geq 0, y \geq 0$  is
- (a) first quadrant
  - (b) Second quadrant
  - (c) Third quadrant
  - (d) Fourth quadrant

## Answers

1. Min  $z = 60$  at  $x = 5, y = 5$   
Max  $z = 180$  at the two corner points  $(0, 20)$  and  $(15, 5)$ .
2. No minimum value
3. Minimum cost = Rs. 1350 at 5 days of A and 3 days of B.
4. 100 kg of fertiliser A and 80 kg of fertilisers B; minimum cost Rs. 1000.
5. Maximum Profit = Rs. 95 with 5 shares of each type.
6. Lamps of type A = 40, Lamps of type B = 20.
7. Fan: 8; Sewing machine: 12, Maximum Profit = Rs. 392.
8. At 25 km/h he should travel  $50/3$  km, at 40 km/h,  $40/3$  km. Maximum distance 30 km in 1 hr.
9. X: 2 units; Y: 6 units; Maximum revenue Rs. 760.
10. Type A: 4; Type B: 3
11. Cup A: 15; Cup B: 30
12. Maximum profit Rs. 1300, No. of belts of type A = 200 No. of belts of type B = 600.
13. No. of first class ticket = 40, No. of second class ticket = 160.
14. Food A: 5 units, Food B: 30 units
15. Maximum interest is Rs. 1160 at  $(2000, 10000)$

**LINEAR PROGRAMMING  
ONE MARK QUESTIONS ANSWER**

1. (d)
2. (c)
3. (b)
4. (a)
5. (c)
6. (c)
7. (b)
8. (a)

## CHAPTER 13

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### PROBABILITY

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#### POINTS TO REMEMBER

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- **Conditional Probability:** If A and B are two events associated with any random experiment, then  $P(A/B)$  represents the probability of occurrence of event A knowing that event B has already occurred.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

$P(B) \neq 0$ , means that the event should not be impossible.

$$P(A \cap B) = P(A \text{ and } B) = P(B) \times P(A/B)$$

- Similarly  $P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/AB)$

$$P(A/S) = P(A), P(A/A) = 1, P(S/A) = 1, P(A^1/B) = 1 - P(A/B)$$

- **Multiplication Theorem on Probability:** If the event A and B are associated with any random experiment and the occurrence of one depends on the other, then

$$P(A \cap B) = P(A) \times P(B/A) \text{ where } P(A) \neq 0$$

- When the occurrence of one does not depend on the other then these event are said to be independent events.

Here  $P(A/B) = P(A)$  and  $P(B/A) = P(B)$

$$P(A \cap B) = P(A) \times P(B)$$

- **Theorem on total probability:** If  $E_1, E_2, E_3, \dots, E_n$  be a partition of sample space and  $E_1, E_2, \dots, E_n$  all have non-zero probability. A be any event associated with sample space S, which occurs with  $E_1$ , or  $E_2$ , ..., or  $E_n$ , then

$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \dots + P(E_n) \cdot P(A/E_n)$$

If A & B are independent then (i)  $A \cap B^c$ , (ii)  $A^c \cap B$  & (iii)  $A^c \cap B^c$  are also independent.

- **Bayes' theorem** : Let S be the sample space and  $E_1, E_2, \dots, E_n$  be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with  $E_1$ , or  $E_2$  or  $\dots, E_n$ , then

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^n P(E_i)P(A/E_i)}$$

- **Random variable** : It is real valued function whose domain is the sample space of random experiment.
- **Probability distribution** : It is a system of number of random variable (X), such that

X:	$X_1$	$X_2$	$X_3, \dots$	$X_n$
P(X):	$P(X_1)$	$P(X_2)$	$P(X_3), \dots$	$P(X_n)$

Where  $P(x_i) > 0$  and  $\sum_{i=1}^n P(x_i) = 1$

- Mean or expectation of a random variables (X) is denoted by E(X)

$$E(X) = \mu = \sum_{i=1}^n x_i P(x_i)$$

- Variance of X denoted by var(X) or  $\sigma_{x^2}$  and

$$Var(X) = \sigma_{x^2} = \sum_{i=1}^n (x_i - \mu)^2 P(x_i) = \sum_{i=1}^n x_i^2 P(x_i) - \mu^2$$

- The non-negative number  $\sigma_x = \sqrt{var(X)}$  is called standard deviation of random variable X.

## ONE MARK QUESTIONS

1. Find  $P(A/B)$  if  $P(A) = 0.4$ ,  $P(B) = 0.8$  and  $P(B/A) = 0.6$
2. Find  $P(A \cap B)$  if  $A$  and  $B$  are two events such that  $P(A) = 0.5$ ,  $P(B) = 0.6$  and  $P(A \cup B) = 0.8$
3. A soldier fires three bullets on enemy: The probability that the enemy will be killed by one bullet is 0.7. What is the probability that the enemy is still alive?
4. If  $P(A) = \frac{1}{2}$ ,  $P(B) = \frac{7}{12}$  and  $P(\text{not } A \text{ or not } B) = \frac{1}{4}$ . State whether  $A$  and  $B$  are independent.
5. Three coins are tossed once. Find the probability of getting at least one head.
6. Find  $P(A/B)$ , if  $P(B) = 0.5$  and  $P(A \cap B) = 0.32$
7. An urn contains 6 red and 3 black balls. Two balls are randomly drawn. Let  $x$  presents the number of black balls. What are the possible value of  $x$ ?
8. A die is tossed thrice. Find the probability of getting an even number at least once.
9. Events  $E$  and  $F$  are such that  $P(\text{Not } E \text{ or Not } F) = 0.25$ . State whether  $E$  and  $F$  are mutually exclusive.
10. Out of 30 consecutive integers 2 are choosen at random. Find the probability so that their sum is odd.
11. If event  $A$  and  $B$  are mutually exclusive and exhaustive events and  $P(A) = \frac{1}{3} P(B)$  then Find  $P(A)$ .
12. A natural number  $x$  is choosen at random from the first hundred natural numbers. Find the probability such that  $x + \frac{1}{x} < 2$
13. A long contains 50 tickets numbered 1, 2, 3, ..... 50 of which five are drawn at random and arranged in ascending order of magnitude. ( $x_1 < x_2 < x_3 < x_4 < x_5$ ). What is the probability that  $x_3 = 30$

## TWO MARK QUESTIONS

1. If A and B are two events such that  $P(A) \neq 0$ , then find  $P(B/A)$  if (i) A is a subset of B (ii)  $A \cap B = \phi$
2. A random variable X has the following probability distribution find K.

X	0	1	2	3	4	5
P(X)	$\frac{1}{15}$	K	$\frac{15K-2}{15}$	K	$\frac{15K-1}{15}$	$\frac{1}{15}$

3. If  $P(A) = \frac{1}{2}$ ,  $P(A \cup B) = \frac{3}{5}$ , and  $P(B) = q$  find the value of q if A and B are (i) Mutually exclusive (ii) independent events.
4. If  $P(A) = \frac{3}{10}$ ,  $P(B) = \frac{2}{5}$  and  $P(A \cup B) = \frac{3}{5}$ , then find  $P(B/A) + P(A/B)$
5. A die is rolled if the out come is an even number. What is the probability that it is a prime?
6. If A and B are two-events such that  $P(A) = \frac{1}{4}$ ,  $P(B) = \frac{1}{2}$  and  $P(A \cap B) = \frac{1}{8}$  Find  $P(\text{not A and not B})$ .



7. The probability that atleast one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, then evaluate  $P(\bar{A}) + P(\bar{B})$ .
8. Three events A, B and C have probabilities  $\frac{2}{5}$ ,  $\frac{1}{3}$ , and  $\frac{1}{2}$ , respectively. If  $P(A \cap C) = \frac{1}{5}$  and  $P(B \cap C) = \frac{1}{4}$ , then find the values of  $P(C/B)$  and  $P(A' \cap C')$ .
9. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.
10. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?
11. Prove that if E and F are independent events, then events E and  $F'$  are also independent.
12. The probability distribution of a discrete random variable X is given below
- |      |               |               |               |                |
|------|---------------|---------------|---------------|----------------|
| x    | 2             | 3             | 4             | 5              |
| P(x) | $\frac{5}{k}$ | $\frac{7}{k}$ | $\frac{9}{k}$ | $\frac{11}{k}$ |
- find the value of K.
13. If  $P(A) = \frac{7}{13}$ ,  $P(B) = \frac{9}{13}$ , and  $P(A \cap B) = \frac{4}{13}$ , then find  $P\left(\frac{A'}{B}\right)$ .
14. In a class XII of a school, 40% of students study mathematics, 30% of the students study Biology and 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, then find the probability that he will be studying Mathematics or Biology.
15. If  $P(A) = 0.4$ ,  $P(B) = 0.8$  and  $P\left(\frac{B}{A}\right) = 0.6$  then find  $P(A \cup B)$ .
16. A die has two faces each with number 1, three faces each with number 2 and one face with number 3. If die is rolled once, then determine probability of not getting 3.
17. A coin is tossed 4 times. Find the mean and variance of the probability distribution of the number of tails.
18. There are 25 tickets bearing numbers from 1 to 25 one ticket is drawn at random. Find the probability that the number on it is a multiple of 5 or 6.

## FOUR MARK QUESTIONS

1. A problem in mathematics is given to three students whose chances of solving it are  $\frac{1}{2}$ ,  $\frac{1}{3}$  and  $\frac{1}{4}$ . What is the probability that the problem is solved?
2. Two aeroplanes X and Y bomb a target in succession. There probabilities to hit correctly are 0.3 and 0.2 respectively. The second plane will bomb only if first miss the target. Find the probability that target is hit by Y plane.
3. Two dice are thrown once. Find the probability of getting an even number on the first die or a total of 8.
4. A and B throw a die alternatively till one of them throws a '6' and wins the game. Find their respective probabilities of winning, if A starts the game.
5. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.
6. Two cards are drawn from a pack of well shuffled 52 cards one by one with replacement. Getting an ace or a spade is considered a success. Find the probability distribution for the number of successes.
7. In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/loses.
8. Suppose that 10% of men and 5% of women have grey hair. A grey haired person is selected at random. What is the probability that the selected person is male assuming that there are 60% males and 40% females?

9. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn. What is the probability that they both are diamonds?
10. If A and B are two independent events such that  $P(\overline{A} \cap B) = \frac{2}{15}$  and  $P(A \cap \overline{B}) = \frac{1}{6}$  then find P(A) and P(B).

### SIX MARK QUESTIONS

1. Bag A contains 4 red, 3 white and 2 black balls. Bag B contains 3 red, 2 white and 3 black balls. One ball is transferred from bag A to bag B and then a ball is drawn from bag B. The ball so drawn is found to be red. Find the probability that the transferred ball is black.
2. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter by other means of transport are respectively  $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$  and  $\frac{2}{5}$ . The probabilities that he will be late are  $\frac{1}{4}, \frac{1}{3}$  and  $\frac{1}{12}$  if he comes by train, bus and scooter respectively but if comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

3. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is six. Find the probability that it is actually a six.
4. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident with scooter, car, and truck driver, is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?
5. Three cards from a pack of 52 cards are lost. One card is drawn from the remaining cards. If drawn card is heart, find the probability that the lost cards were all hearts.
6. A box X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.
7. In answering a question on a multiple choice, a student either knows the answer or guesses. Let  $\frac{3}{4}$  be the probability that he knows the answer and  $\frac{1}{4}$  be the probability that he guesses. Assuming that a student who guesses at the answer will be incorrect with probability  $\frac{1}{4}$ . What is the probability that the student knows the answer, given that he answered correctly?
8. Suppose a girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head. What is the probability that she throws 1, 2, 3 or 4 with the die?
9. In a bolt factory machines, A, B and C manufacture bolts in the ratio 6:3:1. 2%, 5% and 10% of the bolts produced by them respectively are defective. A bolt is picked up at random from the product and is found to be defective. What is the probability that it has been manufactured by machine A?
10. Two urns A and B contain 6 black and 4 white, 4 black and 6 white balls respectively. Two balls are drawn from one of the urns. If both the balls drawn are white, find the probability that the balls are drawn from urn B.

11. Two cards are drawn from a well shuffled pack of 52 cards. Find the mean and variance for the number of face cards obtained.
12. A letter is known to have come from TATA NAGAR or from CALCUTTA on the envelope first two consecutive letters 'TA' are visible. What is the probability that the letter come from TATA NAGAR?
13. Two groups are competing for the position on the Board of Directors of a corporation. The probabilities that first and the second group will win are 0.6 and 0.4 respectively. Further if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.
14. Two numbers are selected at random (without replacement) from positive integers 2, 3, 4, 5, 6, 7. Let X denotes the larger of the two numbers obtained. Find the mean and variance of the probability distribution of X.
15. An urn contains five balls. Two balls are drawn and are found to be white. What is the probability that all the balls are white?
16. Find the probability distribution of the number of doublets in four throws of a pair of dice. Also find the mean and S.D. of his distribution.
17. Three critics review a book. Odds in favour of the book are 5:2, 4:3 and 3:4 respectively for the three critics. Find the probability that the majority are in favour of the book.
18. A box contains 2 Black, 4 White and 3 Red balls. One by one all balls are drawn without replacement and arranged in sequence of drawing. Find the probability that the drawn balls are in sequence of BBWWRRRR.
19. A bag contains 3 White, 3Black and 2 Red balls. 3 balls are successively drawn without replacement. Find the probability that third ball is red.

**Answers**  
**ONE MARK QUESTIONS**

- |                    |  |
|--------------------|--|
| 1. 0.3             | 8. $\frac{7}{8}$   |
| 2. $\frac{3}{10}$  | 9. Not mutually exclusive                                  |
| 3. $(0.3)^3$       | 10. $\frac{15}{29}$  |
| 4. No              | 11. $\frac{1}{4}$  |
| 5. $\frac{7}{8}$   | 12. 0  |
| 6. $\frac{16}{25}$ | 13. $P(E) = \frac{{}^{29}C_2 \cup {}^{20}C_2}{{}^{50}C_5}$ |
| 7. 0, 1, and 2.    |  |

**TWO MARK QUESTIONS**

- |   |                               |
|---|-------------------------------|
| 1. (i) 1 (ii) 0   | 10. $\frac{3}{7}$             |
| 2. $K = \frac{4}{15}$   | 11. –                         |
| 3. (i) $\frac{1}{10}$ (ii) $\frac{1}{5}$  | 12. 32                        |
| 4. $\frac{7}{12}$   | 13. $\frac{5}{9}$             |
| 5. $\frac{1}{3}$  | 14. 0.6                       |
| 6. $\frac{3}{8}$  | 15. 0.96                      |
| 7. 1.1  | 16. $\frac{5}{6}$             |
| 8. $\frac{3}{10}$   | 17. Mean = 2 and variance = 1 |
| 9. A and B are not independent events (s)<br>because $P(A \cap B) \neq P(A) \cdot P(B)$ | 18. $\frac{9}{25}$            |

## FOUR MARK QUESTIONS

1.  $\frac{3}{4}$

2.  $\frac{7}{22}$

3.  $\frac{5}{9}$

4.  $\frac{6}{11}, \frac{5}{11}$

5.  $0.3678$  or  $11C_5(0.4)^5(0.6)^5$

6.

X	0	1	2
P(X)	81/169	72/169	16/169

7.  $-\frac{91}{54}$

8.  $\frac{3}{4}$

9.  $\frac{11}{50}$

10.  $P(A) = \frac{1}{5}, P(B) = \frac{1}{6}$  or  $P(A) = 5/6, P(B) = 4/5$

## SIX MARK QUESTIONS

1.  $\frac{6}{31}$

2.  $\frac{1}{2}$

3.  $\frac{3}{8}$

4.  $\frac{-1}{52}$

5.  $\frac{10}{49}$

6.  $\frac{25}{52}$

7.  $\frac{12}{13}$

8.  $\frac{8}{11}$

9.  $\frac{12}{37}$

10.  $\frac{5}{7}$

11. Mean =  $\frac{6}{13}$  Variance =  $\frac{60}{169}$

12.  $\frac{7}{11}$

13.  $\frac{2}{9}$

14. Mean =  $\frac{17}{3}$  Variance =  $\frac{14}{9}$

15.  $\frac{1}{2}$



16. Mean =  $\frac{2}{3}$  S.D. =  $\frac{\sqrt{5}}{3}$

17.  $\frac{209}{343}$

18.  $\frac{1}{1260}$

19.  $\frac{1}{4}$

**PRACTICE PAPER I**  
**CLASS XII**  
**SECTION A (1 MARKS EACH)**

**General Instructions:**

- (I) All question are compulsory.
- (ii) The question paper consists of 36 questions divided into four sections : A, B, C and D.
- Section A      Comprises of 20 questions of 1 mark each
- Section B      Comprises of 6 questions of 2 mark each
- Section C      Comprises of 6 questions of 4 mark each
- Section D      Comprises of 4 questions of 6 mark each
- (iii) All questions in section A are to be answered in one word, one sentence or as per exact requirement of question.
- (iv) There is no overall choice, however internal choice has been provided in 3 question of section B 3 question of section C and 2 questions of section D, You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculator is not permitted you may ask for logarithmic tables, if required.

1. Find the sum of order and degree of the differential equation  $\frac{d^2y}{dx^2} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^3$
2. Write the smallest reflexive relation on set  $A = \{1, 2, 3, 4, 5\}$ .
3. Write the value of  $\tan^{-1} 2 + \tan^{-1} 3$ .
4. Write the integrating factor of the differential equation  
$$\frac{dx}{dy} + x \tan y - \sec y = 0$$
5. Find ' $\lambda$ '. If the vectors  $\lambda \hat{i} + \hat{j} + 2\hat{k}$ ,  $2\hat{i} - \hat{j} + \lambda \hat{k}$ , and  $\hat{i} + \lambda \hat{j} - \hat{k}$  are coplanar.
6. Find the projection of  $\hat{i} + 3\hat{j} + 7\hat{k}$  on the vector  $2\hat{i} - 3\hat{j} + 6\hat{k}$

7. The probability distribution of a discrete random variable X is given by

X	2	3	4	5
P(x)	$\frac{5}{K}$	$\frac{7}{K}$	$\frac{9}{K}$	$\frac{11}{K}$

Find  $P(x = 3)$

8. If A and B are events such that  $P(A) = \frac{1}{4}$ ,  $P(B) = \frac{1}{2}$  and  $P(A \cap B) = \frac{1}{8}$ , then find  $P(\text{not } A \text{ and not } B)$ .
9. For what value of x, is the following matrix singular?

$$\begin{bmatrix} 3-2x & x+1 \\ 2 & 4 \end{bmatrix}$$

10. If  $[2 \times 3] \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = 0$ , find 'x'

11. Find maximum value of  $z = 2x + 3y$  subject to the constraints  $x + y \leq 4$ ,  $x \geq 0$ ,  $y \geq 0$ .

12. For what value of 'k' the function

$$f(x) = \begin{cases} kx^2, & x \leq 2 \\ 3, & x > 2 \end{cases} \text{ is continuous at } x = 2$$

13. Differentiate  $\sin \sqrt{x} + \cos x^2 \sqrt{x}$  w.r.t. x

14. Evaluate  $\int \sqrt{\frac{x}{1-x^3}} dx$

15. Evaluate  $\int_{-1}^1 (x^7 + \tan^5 x + x + 1) dx$

16. Evaluate  $\int (4 \cot x - 5 \tan x)^2 dx$

17. Find the slope of tangent to the curve  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$  at  $\theta = \frac{\pi}{2}$

18. Show that  $f(x) = 7x - 3$  is strictly increasing on  $\mathbb{R}$ .
19. Of the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.
20. The radius of a balloon is increasing of the rate of 10 cm/sec. At what rate is the surface area of the balloon increasing when the radius is 15 cm?

### SECTION - B (2 MARKS EACH)

21. Prove that  $\cos [\tan^{-1} \{\sin (\cos^{-1} x)\}] = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$ .

OR

Solve  $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\frac{\pi}{2}$

22. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$$

23. If  $y^{1/m} + y^{-1/m} = 2x$ , prove that  $(x^2 - 1)y_2 + xy_1 = m^2y$

OR

If  $y = x \log \left( \frac{x}{a+bx} \right)$ , then prove that  $x^3y_2 = (xy_1 - y)^2$ .

24. Evaluate  $\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$

25. If the magnitude of the vector product of the vector  $\hat{i} + \hat{j} + \hat{k}$  with a unit vector along the sum of vector  $2\hat{i} + 4\hat{j} - 5\hat{k}$  and  $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$  is equal to  $\sqrt{2}$ , then find the value of ' $\lambda$ '.

Or

If  $\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$  and  $\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$ , then express  $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$  such that  $\vec{\beta}_1 \parallel \vec{\alpha}$  and  $\vec{\beta}_2 \perp \vec{\alpha}$ .

26. A problem in Mathematics is given to three students whose probabilities of solving it are  $\frac{1}{2}$ ,  $\frac{1}{3}$  and  $\frac{1}{4}$  respectively. What is the probability that exactly one of them solves it correctly?

**SECTION 'C' (4 Marks Each)**

27. Show that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{x}{x^2 + 1} \quad \forall x \in \mathbb{R}$  is neither one-one nor onto.

Or

Let  $Z$  be the set of all integers and  $R$  be a relation on  $z$  defined as  $R = \{(a, b) : a, b \in z \text{ and } (a - b) \text{ is divisible by } 5\}$ . Prove that  $R$  is an equivalence relation.

28. The tailors A and B are paid ₹225 and ₹300 per day respectively. A can stitch 9 shirts and 6 pants while B can stitch 15 shirts and 6 pants per day. Form a linear programming problem to minimize the labour cost to produce at least 90 shirts and 45 pants. Solve the problem graphically.
29. There are three coins one is a two-headed coin (having head on both faces), another is a biased coin that comes up tails 25% of the times and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows head what is the probability that it was a two-headed coin?
30. Show that the lines  $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$  and  $\frac{8-x}{-7} = \frac{4-y}{-1} = \frac{z-5}{3}$  intersect. Find their point of intersection.

OR

Find the equation of the two lines through the origin which intersect the line

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} \text{ at angles of } \frac{\pi}{3} \text{ each.}$$

31. Prove that  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ , hence evaluate  $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

OR

Evaluate  $\int_1^4 (x^2 - 2x) dx$  as the limit of sums.

32. Solve:

$$x \frac{dy}{dx} \cdot \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0 \text{ given } y = \frac{\pi}{2} \text{ at } x = 1$$

### SECTION-D (6 Marks Each)

33. if  $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & -2 \\ 4 & -6 & -2 \end{bmatrix}$ , find  $A^{-1}$ .

Hence solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = -3, \quad \frac{5}{x} + \frac{4}{y} - \frac{6}{z} = 4, \quad \frac{3}{x} - \frac{2}{y} - \frac{2}{z} = 6$$

34. If the sum of length of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is  $\frac{\pi}{3}$ .

OR

Find the maximum surface area of a circular cylinder that can be inscribed in a sphere of radius 'R'

35. Using integration, find the area of the region bounded by the lines  $x + 2y = 2$ ,  $y - x = 1$  and  $2x + y = 7$ .

36. Find the coordinates of the image of the point (1,3,4) in the plane  $2x - y + z + 3 = 0$

Or

Find the foot of perpendicular drawn from the point P(1,2,3) on the line  $\frac{x-6}{3} = \frac{y-7}{2}$

$= \frac{7-z}{2}$ . Also find the equation of the plane containing the line and the point (1,2,3)

### ANSWERS

#### SECTION-A

- |                   |                                    |                     |
|-------------------|------------------------------------|---------------------|
| 1. 8              | 2. {(1,1) (2,2) (3,3) (4,4) (5,5)} | 3. $\frac{3\pi}{4}$ |
| 4. $\sec y$       | 5. $\lambda = -2$                  | 6. 5                |
| 7. $\frac{7}{32}$ | 8. 3/8                             | 9. $x = 1$          |

10.  $x = 0, -\frac{23}{2}$       11. 12      12.  $k = \frac{3}{4}$
13.  $\frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$       14.  $\frac{2}{3} \sin^{-1}(x^{3/2}) + c$       15. 2
16.  $-16 \cot x + 25 \tan x - 81x + c$       17. 1
19.  $9.72 \pi \text{ cm}^3$       20.  $1200 \pi \text{ cm}^2/\text{sec.}$

#### SECTION-B

21. OR  $0, \pm \frac{1}{2}$       24.  $\frac{-1}{14} \tan^{-1} \frac{x}{2} + \frac{8}{35} \tan^{-1} \frac{x}{5} + c$
25.  $\lambda = 1$  OR  $\vec{B} = -\frac{1}{5}(3\hat{i} + 4\hat{j} + 5\hat{k}) + \frac{1}{5}(13\hat{i} + 9\hat{j} - 15\hat{k})$

#### SECTION : C

26.  $\frac{11}{24}$
28. A work for 5 days and B works for 3 days Minimum labour cost = ₹ 2025
29.  $\frac{4}{9}$       30.  $(1, 3, 2)$  Or  $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$  and  $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$
31.  $\frac{\pi^2}{4}$  Or 6      32.  $\log x = \cos\left(\frac{y}{x}\right)$

#### SECTION : D

33.  $A^{-1} = \frac{1}{152} \begin{bmatrix} 20 & 8 & 22 \\ 2 & 16 & -13 \\ 34 & -32 & 7 \end{bmatrix}$ ,  $x = 1, y = -1, z = -2$
34.  $\pi R^2 (\sqrt{5} + 1)$  square units      35. 6 square units
36.  $(-3, 5, 2)$  Or  $(3, 5, 9), 18x - 22y + 5z + 11 = 0$

**PAPER II CLASS XII**  
**PRACTICE SECTION A (1 MARKS EACH)**

1. If  $R = \{(x, y) : x + 2y = 8\}$  is a relation on  $N$ , write the range of  $R$ .
2. If  $\sin(\sin^{-1} \frac{3}{5} + \cos^{-1} x) = 1$  then find value of  $x$ .
3. How many matrices of order  $2 \times 2$  are possible with entry  $2 \times 2$ .
4. If  $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ , find  $|A^{-1}|$
5. If  $y = |x|$ , then find  $dy/dx$ .
6. If  $y = \sin x + \tan^{-1}(1)$ , find  $dy/dx$
7. Find the minimum value of  $\sin x \cos x$ .
8. Which of the following function is decreasing on  $(0, \frac{\pi}{2})$ ?  
a)  $\sin 2x$     b)  $\cos 3x$     c)  $\tan x$     d)  $\cos 2x$
9. The curves  $y = ae^x$  and  $y = be^{-x}$  cut orthogonally if  
a)  $a=b$     b)  $ab=-b$     c)  $ab=1$     d)  $ab=2$
10. Evaluate :  $\int \frac{dx}{1-\sin^2 x}$
11. Evaluate :  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$
12. Evaluate :  $\int \frac{\sin x}{\sin 2x} \, dx$
13. The degree of  $\frac{dy}{dx} + \cos y = 0$  is not defined true or false?
14. Write the order of the following differential equation :  $\sqrt{1 + \frac{dy}{dx}} = \left(\frac{d^2 y}{dx^2}\right)^{1/3}$



15. write integrating factor of the following differential equation:-

$$\frac{dx}{dy} + x \cos y + \sin y$$

16. If  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$  are three mutually perpendicular vectors, then find the value of  $\hat{j} \cdot (\hat{k} \times \hat{i})$

17. What is the perpendicular distance of plane  $2x - y + 3z = 10$  from origin?

18. Define an objective function.

19. Find  $P(A/B)$  if  $P(A) = 0.4$ ,  $P(B) = 0.8$  and  $P(B/A) = 0.6$

20. Three coins are tossed once. Find the probability of getting at least one head.

### Section B (2 Mark Each)

21. Show that the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2 + 1$  is not invertible.

22. Using properties of determinants, show that

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 0$$

OR

Prove that the diagonal elements of a skew - symmetric matrix are all zero.

23. Find the domain of the continuity of  $f(x) = \sin^{-1} x - [x]$

24. If  $y = x^x$ , find  $\frac{d^2y}{dx^2}$

OR

if  $x = 2at^2$ ,  $y = at^4$ , find  $\frac{d^2y}{dx^2}$

25. Find the equation of the plane passing through the point  $(-2, 1, -3)$  and making equal intercept on the co ordinate axes.

26. Two balls are drawn at random from a bag containing 6 red and 4 green balls, find the probability that both ball are of same colour.

### Section C (4 Mark Each)

27. Show that the relation  $R$  defined by  $(a,b) R (c,d) \iff a+b = b+c$  on the set  $N \times N$  is an equivalence relation.

Or

Let  $f : N \rightarrow R$  be a finction defined as  $f(x) = 4x^2 + 12x + 15$  show that  $f : N \rightarrow S$ , where  $S$  is the range of  $f$ , is invertible. Also find the inverse of  $f$ .

28. Evaluate :  $\int \frac{\cos(x+a)}{\cos(x-a)} dx$

Or

Evaluate :  $\int \frac{x}{x^4 + x^2 + 1} dx$

29. Solve the following differential equation:

$$(x^3 + y^3) dx + (x^2y + xy^2) dy$$

30. Decompose the vector  $6\hat{i} - 3\hat{j} - 6\hat{k}$  into vectors which are parallel and perpendicular to the vector  $\hat{i} + \hat{j} + \hat{k}$

31. A company produces two types of belts A and B. Profits on these belts are ₹ 2 and ₹ 1.50 per belt respectively. A belt of type A requires as much time as belt of type B. The company can produce at most 1000 belts of type B per day. Material for 800 belts per day is available. At most 400 buckles for belts of type A and 700 foe type B are available per day. How much belts of each type should the company produce so as to maximize the profit?

32. Two urns A and B contain 6 black and 4 white, 4 black and 6 white balls respectively. Two balls are drawn from one of the urns. If both the balls drawn are white, find the probability that the balls are drawn from urn B.

### Section D (6 Mark Each)

33. If  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ , find  $A^{-1}$  and use it to solve the system of equations :

$$x+2y+z=4, -x+y+z=0, x-3y+z=2$$

34. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius  $r$  is  $\frac{2r}{\sqrt{3}}$

OR

Find the area of greatest rectangle that can be inscribed in an ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

35. Find the area of the region.

$$\{(x, y) : y \leq 8, 6x, x^2 + y^2 \leq 6\}$$

Or

Using integration, find the area of the triangle whose vertices are  $(-1, 0)$ ,  $(1, 3)$  and  $(3, 2)$ .

36. Find the foot of perpendicular from the point  $2\hat{i} - \hat{j} + 5\hat{k}$  on the Line.

$$\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda (10\hat{i} - 4\hat{j} - 11\hat{k})$$
 Also find the length of the perpendicular.

### ANSWERS OF PRACTICE PAPER-II

1.  $\{1, 2, 3\}$
2.  $x = 3/5$
3. 16
4.  $1/10$
5.  $\frac{x}{|x|}$
6.  $\cos x$
7.  $-\sqrt{2}$
8. (d)
9. (c)
10.  $\tan x$
11. 0
12.  $\frac{1}{2} \log |\sec x + \tan x| + c$
13. True
14. 2
15.  $e^{\sin y}$
16. 1
17.  $10\sqrt{14}$
19. 0.3
20.  $7/8$
23.  $(-1, 0) \cup (0, 1)$
24.  $x^x \left[ (1 + \log x)^2 + \frac{1}{x} \right]$   
OR  
 $\frac{1}{2a}$
25.  $x + y + z = -4$
26.  $7/15$
27.  $f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}$
28.  $x \cos 2a - \sin 2a \log [\sec (x-a)] + c$   
Or  
 $\frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x^2+1}{\sqrt{3}} \right) + C$
29.  $-y = x \log [c(x-y)]$
30.  $(-i - j - k) + (7i - 2j - 5k)$
31. Max. profits = ₹ 1300  
No. of belts type A = 200  
No. of belts type B = 600
32.  $5/7$
33.  $A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$   
 $x = \frac{9}{5}, y = \frac{2}{5}, z = \frac{7}{5}$
34.  $2ab$  sq. units.
35.  $\frac{32\pi - 4\sqrt{3}}{3}$  sq. units. Or 4 sq. units
36.  $(1, 2, \sqrt[3]{3}), \sqrt{14}$

## **PRACTICE PAPER III**

### **CLASS XII**

- (I) All question are compulsory.
- (ii) The question paper consists of 36 questions divided into four sections : A, B, C and D.

Section A	Comprises of 20 questions of 1 mark each
Section B	Comprises of 20 questions of 2 mark each
Section C	Comprises of 20 questions of 4 mark each
Section D	Comprises of 20 questions of 6 mark each

- (iii) All questions in section A are to be answered in one word, one sentence or as per exact requirement of question.
- (iv) There is no overall choice, however internal choice has been provided in 2 question of section B 2 question of section C and 2 questions of section D, You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculator is not permitted you may ask for logarithmic tables, if required.

## Questions Section-A

Question numbers 1 to 20 carry 1 mark each.

1. Given  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Write the value of AB.

2. What is order and degree (if defined) of differential equation.

$$\frac{d^2y}{dx^2} + x \left( \frac{dy}{dx} \right)^2 = 2x^2 \cdot \log \left( \frac{d^2y}{dx^2} \right)$$

3. The angle between the line  $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$  and the plane  $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$  is \_\_\_\_\_.

4. The co-ordinate of the joint, where the line  $\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5}$  cuts the yz plane is \_\_\_\_\_.

5. If  $y = \sin^{-1}x + \cos^{-1}x$ , write  $\frac{dy}{dx}$

6. If  $f(x) = x + 1$ , write the value of  $\frac{d}{dx} f \circ f(x)$ .

7. If A is a square matrix of order 3 with  $|A|=4$ . Then write all value of  $|2A|$ .

8. If event A and B are mutually exclusive and exhaustive events and  $P(A) = \frac{1}{3}$ ,  $P(B)$ , then  $P(A)$  is equal to \_\_\_\_\_.

9. In which quadrant the bounded region for in equations  $x+y \leq 1$  and  $x-y \leq 1$  is situated?

A) I, II      B) I, III      C) II, III      D) All four quadrants.

10. Write the derivative of  $e^x$  wrt.  $\sqrt{x}$
11. Find the differential equation representing the family of curves  $y = a \cdot e^{2x} + 5$ , where  $a$  is an arbitrary constant.
12. Write the maximum value of  $f(x) = \frac{\log x}{x}$  if it exists.
13. Evaluate :  $\int \frac{1 + \cos x}{x + \sin x} dx$ .
14. Evaluate :  $\int_2^3 3^x dx$
15. Find the integrating factor of  $x \frac{dy}{dy} + 2y = x \cos x$
16. Write value of  $(\hat{k} \times \hat{j}) \hat{i} + \hat{j} + \hat{k}$
17. Evaluate :  $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^3 x dx$
18. Find :  $\int \frac{2x}{(x^2+1)(x^2+3)} dx$ .
19. Slope of tangent to the curve  $y = x^2 + x + 1$  at  $x = 1$  is \_\_\_\_\_.
20. Write derivative of

$$\cos^{-1} \left( \frac{\sin x + \cos x}{\sqrt{2}} \right), \text{ w.r.t } x - \frac{\pi}{4} < x < \frac{\pi}{4}$$

## Section-B

Question numbers 21 to 26 carry 2 mark each.

21. If  $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ , show that  $(A-2I)(A-3I) = 0$ .

22. Show by examples that the relation  $R$  is  $R$ , defined by  $R = \{(a,b) : a \leq b^3\}$  is neither reflexive nor transitive.

OR

Show that the functions  $f: R \rightarrow R$ , given by  $f(x) = \cos x \forall x \in R$  is neither one-one nor onto.

23. Find  $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos 2x} dx$ .

Or

Find  $\int \frac{x-3}{(x-11)^3} e^x dx$

24. Find  $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$ .

25. Find the volume of a cuboid whose edges are given by

$$-3\hat{i} + 7\hat{j} + 5\hat{k}, -5\hat{i} + 7\hat{j} - 3\hat{k} \text{ and } -7\hat{i} - 5\hat{j} - 3\hat{k}$$

26. Find the probability distribution of  $x$ , the number of heads in a simultaneous toss of two coins.

## Section-C

Question numbers 27 to 32 carry 4 mark each.

27. If  $\tan^{-1} x - \cot^{-1} x = \tan^{-1} \left( \frac{1}{\sqrt{3}} \right)$ ,  $x > 0$  find the value of  $x$  and hence find the value of  $\sec^{-1} \left( \frac{2}{x} \right)$



28. The scalar product of the vector  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$  with a unit vector along the sum of if the vectors  $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$  and  $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$  is equal to 1 find the value of  $\lambda$  and hence find the unit vector along  $\vec{b} + \vec{c}$ .

29. If  $(\sin x)^2 = x + y$  find  $\frac{dy}{dx}$

Find  $\frac{dy}{dx}$  if  $y = \sin^{-1} \left[ \frac{2^{x+1}}{1+4^x} \right]$

OR

If  $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$ , find  $A^{-1}$

30. If  $y = e^x [\sin x + \cos x]$ , prove that

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0$$

31. Minimize  $z = 6x + 3y$

subject to the constraints

$$4x + y \geq 80, x + 5y \geq 115, 3x + 2y \leq 150, x \geq 0, y \geq 0$$

OR

The corner points of the feasible region determined by the system of linear constraints are (0,3), (1,1), and (3,0). Let  $z = px + qy$ , where,  $P, q > 0$ . Find the condition in  $p$  and  $q$ , so that minimum of  $z$  occurs at (3,0) and (1,1).

32. If  $A$  and  $B$  are two independent events such that  $P(A) = \frac{1}{4}$ ,  $P(B) = \frac{1}{2}$  and  $P(A \cap B) = \frac{1}{8}$  then find  $P(\text{not } A \text{ and not } B)$

## Section-D

Question numbers 33 to 36 carry 6 mark each.

33. Using matrix method, solve the following system of equations:

$$3x-2y+3z=8$$

$$2x+y-z=1$$

$$4x-3y+2z=4$$

34. Find the vector and cartesian equations of the plane passing through the points  $(2,2,-1)$ ,  $(3,4,2)$  and  $(7,0,6)$ . Also find the vector equation of a plane passing through  $(4,3,1)$  and parallel to the plane obtained above.

OR

Find the equation of the line passing through  $(2,-1, 2)$  and  $(5,3,4)$  and of the plane passing through  $(2,0,3)$ ,  $(1,1,5)$  and  $(3,2,4)$ . Also find their point of intersection.

35. Using integration find the area of triangle whose vertices are  $(1,0)$   $(2,2)$  and  $(3,1)$  Or  
Using integration, find the area of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

36. Show that height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to radius of its base.

### Practice Paper 3 Answer Making Scheme

S.No.	Value Points	Marks
1.	$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	[1]
2.	Order =2, Degree not defined	[½ + ½]
3.	$\sin \theta = \frac{[(\hat{i} + \hat{j} + \hat{k}) \cdot (3\hat{i} - \hat{j} + 2\hat{k})]}{\sqrt{1+1+1} \sqrt{9+1+4}}$ $\theta = \sin^{-1} \left[ \frac{4}{\sqrt{42}} \right]$	[½ + ½]
4.	<p>Any point on <math>\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5} = \lambda</math> is</p> <p><math>(\lambda-2, 3\lambda+5, 5\lambda-1)</math>, the line cuts yz plane at <math>\lambda-2=0</math> ie <math>\lambda=2</math></p> <p>Hence required point is (0,11,9)</p>	[1]
5.	$y = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$	[1]
6.	$\frac{dy}{dx} [f \circ f(x)] = 1$	[1]
7.	$  -2A   = (-2)^3  A $ $= -8 \times 4 = -32.$	[1]

S.No.	Value Points	Marks
8.	<p>Because event A and B are mutually exclusive and exhaustive.  <math>A \cup B = \text{sample space } S</math>  <math>\therefore P(A \cup B) = P(S) = 1</math>  <math>\Rightarrow P(A) + P(B) - P(A \cap B) = 1</math>  <math>\Rightarrow P(A) + 3P(A) - 0 = 1</math>  <math>\therefore</math> event A and B are mutually exclusive (given)  <math>\Rightarrow A \cap B = \phi</math>  <math>P(A \cap B) = 0</math>  <math>\Rightarrow 4P(A) = 1 \Rightarrow P(A) = \frac{1}{4}</math></p>	<p><math>f(n) = \sin 3(-3) = -\sin x^3 = -f(x)</math>  <math>[\frac{1}{2} + \frac{1}{2}]</math>    <math>[1]</math></p>
9.	All four quadrant.	$[1]$
10.	$e^x 2\sqrt{x}$	$[1]$
11.	$\frac{dy}{dx} = 2ae^{2x} \Rightarrow \frac{dy}{dx} = 2(y-5)$ $\Rightarrow \frac{dy}{dx} - 2y + 10 = 0$	$[\frac{1}{2} + \frac{1}{2}]$
12.	$\frac{1}{e}$	$[1]$
13.	<p>Let <math>I = \int \frac{(1 + \cos x)}{x + \sin x} dx</math>, now put <math>x + \sin x = t</math>  <math>\therefore I = \int \frac{1}{t} dt = \log  t  + c \Rightarrow (1 + \cos x) du = dt</math>  <math>= \log  x + \sin x  + C</math>  <math>\quad \quad \quad   \because t = x + \sin x</math></p>	$[\frac{1}{2} + \frac{1}{2}]$
14.	$\int_2^3 3^x dx = \left( \frac{3^x}{\log 3} \right)_2^3 = \frac{1}{\log 3} \cdot [3^x]_2^3 = \frac{1}{\log 3} [3^3 - 3^2]$ $= \frac{1}{\log 3} (27 - 9) = \frac{18}{\log 3}$	$[\frac{1}{2} + \frac{1}{2}]$
15.	$x^2$	$[1]$
16.	-1	$[1]$

S.No.	Value Points	Marks
17.	<p>Let <math>I = \int_{-\pi/3}^{\pi/2} \sin^3 x \, dx</math></p> <p>Here <math>f(x) = \sin^3 x</math></p> <p><math>\therefore f(-x) = \sin^3(-x) = -\sin^3 x = -f(x)</math></p>	
18.	<p>Put <math>x^2 = t \Rightarrow 2x \, dx = dt</math> and then <math>I = \int \frac{dt}{(t+1)(t+3)}</math></p> <p>Ans <math>\frac{1}{2} \log \frac{x^2+1}{x^2+3} + c</math></p>	$[\frac{1}{2} + \frac{1}{2}]$
19.	3.	[1]
20.	1	[1]
<b>Section B</b>		
21.	$(A-2I)(A-3I) = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$	[1 + 1]
22.	<p><math>a = \frac{1}{2}</math>, not reflexive <math>\frac{1}{2} \leq \frac{1}{2}</math> so</p> <p>R is not Reflexive</p> <p><math>a=9, b=4, c=2</math>, not transitive</p> <p>OR</p> <p>we have function <math>f : R \rightarrow R</math>, defined by</p> <p><math>f(x) = \cos x</math></p> <p><math>f(0) = \cos 0 = 1</math></p> <p><math>f(2\pi) = \cos 2\pi = 1</math></p> <p>so <math>f</math> is not one-one</p> <p>also range <math>f = [-1, 1] \neq R</math></p> <p>hence <math>f</math> is not onto</p>	<p>[1 + 1]</p> <p>[1]</p> <p>[1]</p>

S.No.	Value Points	Marks								
23.	$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cdot \cos^2 x} = \int (\sec x + \tan x + \operatorname{cosec} x \cdot \cot x) dx$ $= \sec x - \operatorname{cosec} x + C$ <p style="text-align: center;">OR</p> $\int \frac{x-3}{(x-1)^3} e^x dx = \int e^x [(x-1)^2 - 2(x-1)^{-3}] dx$ $= e^x [x-1]^{-2} + C \text{ or } \frac{e^x}{(x-1)^2} + C$	<p>[1 + 1]</p> <p>[1]</p> <p>[1]</p>								
24.	Put $\tan x = t \Rightarrow \sec^2 x = dt$ $I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 1}} dx = \int \frac{dt}{\sqrt{t^2 + 4}} dx = \log \left  t + \sqrt{t^2 + 4} \right  + C$ $= \log \left  -\tan x + \sqrt{\tan^2 x + 4} \right  + C$	<p>[1/2]</p> <p>[1/2]</p> <p>[1/2]</p>								
25.	$\vec{a} \cdot (\vec{b} \cdot \vec{c}) = (-3\hat{i} + 7\hat{j} + 5\hat{k}) \cdot (-5\hat{i} + 7\hat{j} - 3\hat{k})$ $= \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & 3 \end{vmatrix} = -264$ $=  -264  = 264 \text{ cubic units}$	<p>[1]</p> <p>[½ + ½]</p>								
26.	Let $x$ = no of heads in simultaneous toss of two coins <table><tr><td><math>x</math> :</td><td>0</td><td>1</td><td>2</td></tr><tr><td><math>P(x)</math> :</td><td>1/4</td><td>1/2</td><td>1/4</td></tr></table>	$x$ :	0	1	2	$P(x)$ :	1/4	1/2	1/4	<p>[2]</p>
$x$ :	0	1	2							
$P(x)$ :	1/4	1/2	1/4							
27.	Given $\tan^{-1} x - \cot^{-1} x = \tan^{-1} \left( \frac{1}{\sqrt{3}} \right), x > 0$ $\Rightarrow \tan^{-1} x - \left( \frac{\pi}{2} - \tan^{-1} x \right) = \frac{\pi}{6}$ $\Rightarrow 2 \tan^{-1} x = \frac{\pi}{2} + \frac{\pi}{6} \Rightarrow \tan^{-1} x = \frac{\pi}{3}$ $\Rightarrow x = \tan \left( \frac{\pi}{3} \right) = \sqrt{3} \therefore \sec^{-1} \frac{2}{x} = \frac{\pi}{6}$	<p>[1]</p> <p>[1]</p> <p>[1 + 1]</p>								

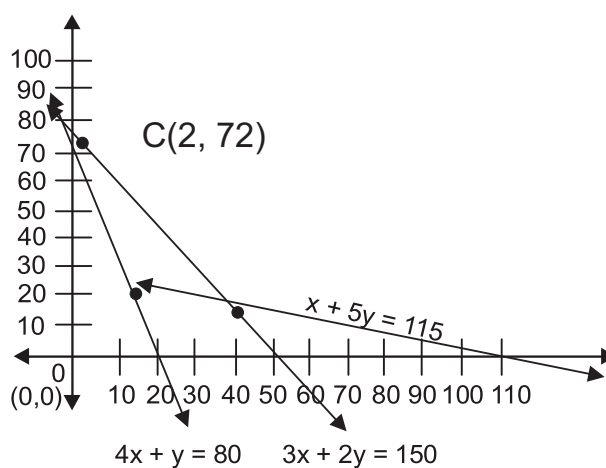
S.No.	Value Points	Marks
28.	$\vec{a} \cdot \frac{(\vec{b} + \vec{c})}{ \vec{b} + \vec{c} } = 1$ $\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot [(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}] = \sqrt{(2 + \lambda)^2 + 36 + 4}$ $\Rightarrow \lambda + 6 = \sqrt{(2 + \lambda)^2 + 40}$ <p>Squaring to get</p> $\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44 \Rightarrow \lambda = 1$ <p><math>\therefore</math> Unit vector along <math>(\vec{b} + \vec{c})</math> is</p> $\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$	<p>[1]</p> <p>[1½]</p> <p>[½]</p> <p>[1]</p>
29.	$(\sin x)^y = (x + y) \Rightarrow y \cdot \log \sin x = \log (x + y)$ <p>Differentiating w.r.t. x we get</p> $y \cdot \cot x + \log \sin x \cdot \frac{dy}{dx} = \frac{1}{x + y} \left( 1 + \frac{dy}{dx} \right)$ $\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{x + y} - y \cot x}{\log \sin x - \frac{1}{x + y}} = \frac{1 - y(x + y) \cot x}{(x + y) \log \sin x - 1}$ <p>Or</p> $y = \sin^{-1} \left[ \frac{2 \cdot 2^x}{1 + (2^x)^2} \right] = \sin^{-1} \left[ \frac{2t}{1 + t^2} \right], \text{ where } t = 2^x$ $\Rightarrow y = 2 \tan^{-1} t, \frac{dy}{dt} = \frac{2}{1 + t^2} \text{ and } \frac{dt}{dx} = 2^x \cdot \log 2$ $\Rightarrow \frac{dy}{dx} = \frac{2}{1 + t^2} \cdot 2^x \cdot \log 2 = \frac{2^{n+1} \cdot \log 2}{1 + 4^x}$	<p>[1]</p> <p>[1½]</p> <p>[1½]</p> <p>[1]</p> <p>[1]</p> <p>[1]</p>
30.	$y = e^x [\sin x + \cos x]$ $\frac{dy}{dx} = e^x [\cos x - \sin x] + e^x \cdot [\sin x + \cos x]$ $= e^x [\cos x - \sin x] + y$	<p>[1]</p> <p>[1/2]</p>

$$\frac{d^2y}{dx^2} = e^x[-\sin x - \cos x] + [\cos x - \sin x] \cdot e^x + \frac{dy}{dx} \quad [1 \frac{1}{2}]$$

$$= -y + \frac{dy}{dx} - y + \frac{dy}{dx} \quad [1/2]$$

$$\frac{d^2y}{dx^2} = -2\frac{dy}{dx} + 2y = 0 \quad [1]$$

31.



Corner Points	Value of Z
A (40, 15)	285
B (15, 20)	150 (Min)
C (2, 72)	228

**Ans.** Minimum  $z = 150$ , when  $x = 15$ ,  $y = 20$

OR

Given Corner points of feasible region are (0,3), (1,1) and (3,0) the Value of objective function at these corner points are given below:

[1]

Corner points	$z = px + qy$
(0, 3)	$z = p \times 0 + q \times 3 = 3q$
(1, 1)	$z = p \times 1 + q \times 1 = p + q$
(3, 0)	$z = p \times 3 + q \times 0 = 3p$

[3]



32. Hint

$$P(\text{not A and not B}) = P(A') \times P(B') \quad [1]$$

$\therefore$  A and B are independent events]

$$= [1-P(A)] [1-P(B)]$$

$$= \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{2}\right)$$

$$= \text{Ans } \frac{3}{8} \quad [3]$$

### SECTION-D

$$33. \text{ Let } A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}; x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

system of equation in Matrix form;

$$A.X = B$$

$$\text{Solution Matrix, } X = A^{-1} B \quad [1]$$

$$|A| = 3(2-3) + 2(4+4) + 3(6-4) = -17 \quad [1]$$

co-factors of Matrix A:

$$\begin{array}{lll} C_{11} = -1, & C_{12} = -8, & C_{13} = -10 \\ C_{21} = -5, & C_{22} = -6, & C_{23} = -1, \\ C_{31} = -1, & C_{32} = 9, & C_{33} = 7, \end{array} \quad [2]$$

$$\therefore A^{-1} = \frac{-1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \quad [1\frac{1}{2}]$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow x = 1, y = 2, z = 3 \quad [1\frac{1}{2}]$$

34. Equation of plane is

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0 \quad [2]$$

$$\Rightarrow 5x + 2y - 3z = 17 \text{ (cartesian equation)}$$

vector equation is

$$\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17 \quad \text{Equation of required plane is} \quad [1]$$

$$\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = (4\hat{i} + 3\hat{j} + \hat{k}) \cdot (5\hat{i} + 2\hat{j} - 3\hat{k})$$

$$\Rightarrow \vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 23 \quad [1]$$

OR

$$\text{Equation of line : } \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} \quad [1]$$

$$\text{Equation of plane: } \begin{vmatrix} x-2 & y & x-3 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0 \quad [1]$$

$$\Rightarrow -3(x-2) + 3y - 3(z-3) = 0$$

$$\Rightarrow x - y + z - 5 = 0$$

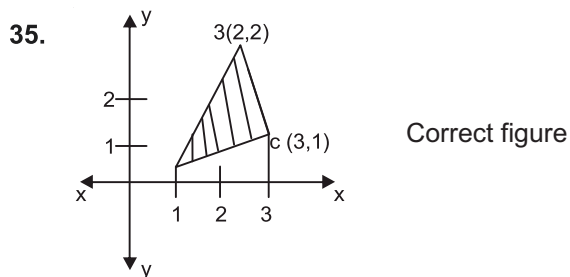
$$\text{General point on line: } \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = K \text{ (say)} \quad [1]$$

$$\text{is } p(3K+2, 4K-1, 2+2); \text{ putting in Eqn. of plane} \quad [1]$$

$$\text{we get } 3K+2-4K+1+2K+2=5 \quad [1]$$

$$\Rightarrow K = 0$$

$$\therefore \text{Point of intersection is } (2, -1, 2) \quad [1]$$



$$\text{Equation of line AB : } y = 2(x-1)$$

$$\text{Equation of line BC : } y = 4 - x \quad [1 \frac{1}{2}]$$

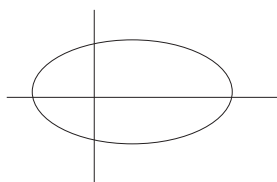
$$\text{Equation of line AC : } y = \frac{1}{2}(x-1)$$

$$\text{ar } (\Delta ABC) = 2 \int_1^2 (x-1) dx + \int_2^3 (4-x) dx - \frac{1}{2} \int_1^3 (x-1) dx \quad [1\frac{1}{2}]$$

$$= (x-1)^2 \Big|_1^2 - \frac{1}{2} (4-x^2) \Big|_2^3 - \frac{1}{4} (x-1)^2 \Big|_1^3 \quad [1\frac{1}{2}]$$

$$= 1 + \frac{3}{2} - 1 = \frac{3}{2} \text{ sq. units}$$

Or



$$\text{Area of ellipse} = 4 \left[ \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx \right] \quad [3]$$

$$= 4 \left[ \left( \frac{b}{a} \left( \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) \right) \right]_0^a \quad [2]$$

$$= 4 \frac{b}{a} \left( \frac{\pi a^2}{4} \right) = \pi ab \quad [1]$$

**36.** Let given surface area of open Cylinder be S

Then  $S = 2\pi rh + \pi r^2$

$$\Rightarrow h = \frac{S - \pi r^2}{2\pi r} \quad [1]$$

$$\text{Volume} = \pi r^2 h \quad [\frac{1}{2}]$$

$$V = \pi r^2 \left[ \frac{S - \pi r^2}{2\pi r} \right] = \frac{1}{2} [Sr - \pi r^3] \quad [1]$$

$$\frac{dv}{dr} = \frac{1}{2} [S - 3\pi r^2] \quad [\frac{1}{2}]$$

$$\frac{dv}{dr} = 0 \quad [1]$$

$$\Rightarrow S = 3\pi r^2 \text{ or } 2\pi rh + \pi r^2 = 3\pi r^2 \quad [1]$$

$$\Rightarrow 2\pi rh = 2\pi r^2$$

$$\Rightarrow h = r$$

$$\frac{d^2v}{dr^2} = -6\pi r < 0$$

For vol to be maximum height = radius. [1]